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O. Neugebauer M. H. Stone O. Veblen

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TABLE OF CONTENTS

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Mathematical Reviews

Vol. 7, No. 1 JANUARY, 1946 Pages 1-44

ALGEBRA

Abstract Algebra

Day, Mahlon M. Arithmetic of ordered systems. Trans. Amer. Math. Soc. **58**, 1–43 (1945). [MF 12771]

Let a set Y_x correspond to each index x of a class X, and let ρ be any binary relation defined on X and all Y_x. Define the generalized sum $S(X, Y_x)$ as the set of pairs (x, y) with $x\epsilon X$ and $y\epsilon Y_x$, making $(x, y)\rho(x', y')$ mean $x\rho x'$ if $x\neq x'$ and $y\rho y'$ if $x=x'$. Define the generalized product $P(X, Y_x)$ as the set of functions f which assign to each $x\epsilon X$ a value $f(x)\epsilon Y_x$, letting $f\rho g$ mean that for each x with $f(x)\neq g(x)$, x' exists with $x'\rho x$ and $f(x)\rho g(x)$. If X is the unordered pair $\{1, 2\}$, then $S(X, Y_x)$ is the cardinal sum and $P(X, Y_x)$ the cardinal product of Y_1 and Y_2; if X is the ordered pair $\{1, 2\}$, then $S(X, Y_x)$ is the ordinal sum and $P(X, Y_x)$ the ordinal product of Y_1, Y_2; if every $Y_x = Y$, then $P(X, Y_x)$ is the ordinal power ${}^XY$. These definitions are not new [cf. G. Birkhoff, Duke Math. J. **3**, 311–316 (1937), in particular, pp. 315–316; Whitehead and Russell, Principia Mathematica, vol. 2, Cambridge University Press, 1912, §§162, 172].

The author remarks that generalized products of partially ordered sets need not be transitive, contrary to an erroneous statement by the reviewer [Duke Math. J. **9**, 283–302 (1942), in particular, p. 291]. He shows that most of the laws of "arithmetic" which the reviewer [loc. cit.] proved for partially ordered systems apply more generally to all reflexive relations; these include most of the laws of transfinite arithmetic. He sharpens some of these laws. He calls a relation ρ k-transitive if, for every chain $x, s_1, \cdots, s_n, y$ such that $x\rho s_1\rho s_2\rho \cdots \rho s_n\rho y$, there exist $t_1, \cdots, t_k$ such that $x\rho t_1\rho \cdots \rho t_k\rho y$; thus transitivity in the usual sense is 0-transitivity. He proves that, if X is m-transitive and every Y_x is k-transitive, then $S(X, Y_x)$ is n-transitive, where $n=\sup(m+1, k)$; if X is partially ordered, $n=k$. If X is partially ordered and the Y_x are all transitive, then $P(X, Y_x)$ is 1-transitive; if X and the Y_x are all transitive, then $P(X, Y_x)$ is 2-transitive. Eight conditions are given which together are necessary and sufficient for $P(X, Y_x)$ to be 0-transitive; three conditions for $P(X, Y_x)$ to be a partial ordering; four conditions for it to be a lattice. Many other interesting results are included. *G. Birkhoff.*

Dilworth, R. P. Lattices with unique complements. Trans. Amer. Math. Soc. **57**, 123–154 (1945). [MF 11914]

It has been widely conjectured that every lattice with unique complements is a Boolean algebra. This conjecture is disproved; the principal tool used is the technique of "free algebras," by means of which many other results are established.

Let P be any partially ordered set. Then there exists a free algebra $L(P)$ among the lattices generated by P and preserving $\cap$ and $\cup$ whenever these exist in P; there exists a free algebra $O(P)$ among the lattices with unary operator $a\to a^*$ generated similarly by P; there exists a free algebra $N(P)$ among the lattices with operator $a\to a^*$ satisfying $(a^*)^*=a$ and generated similarly by P; there exists a free algebra $M(P)$ among the lattices with unique complements (that is, with 0, I, and operator $a\to a^*$ satisfying $(a^*)^*=a$, $a\cap a^*=0$, $a\cup a^*=I$, and $a\cap x=0$ and $a\cup x=I$ imply $x=a^*$). Moreover, P is contained isomorphically in $L(P)$, $O(P)$, $N(P)$ and $M(P)$, with preservation (as stated) of $\cap$ and $\cup$ whenever they exist in P; $M(P)$ is defined as a subset of $N(P)$, with preservation of $\cap$ and $\cup$ except when $x\cap y=0$ or $x\cup y=I$; $N(P)$ is a sublattice of $O(P)$, consisting of all polynomials of which no part has the form $(a^*)^*$. In case P is an unordered set of n elements, we get the free algebras with unrestricted generators of the type described; $L(3)$ contains $L(d)$ as a sublattice (d=countable infinity; this was known); $O(1)$ contains $O(d)$ as a subalgebra; $N(1)$ contains $N(d)$ as a subalgebra; $M(2)$ contains $M(d)$ as a subalgebra; these results are "best possible." In case P is a lattice, we see that every lattice is isomorphic with a sublattice of a lattice with unique complements. Since every sublattice of a Boolean algebra is distributive, it follows that not every lattice with unique complements can be a Boolean algebra. *G. Birkhoff* (Cambridge, Mass.).

Foster, Alfred L. The idempotent elements of a commutative ring form a Boolean algebra; ring-duality and transformation theory. Duke Math. J. **12**, 143–152 (1945). [MF 12076]

Let $a\to a^*$ be a unary operation in an algebraic system A, such that $(a^*)^*=a$. If to each operation $f(x_1, \cdots, x_n)$ defined in A we make correspond the operation f^* defined by $f^*(x_1, \cdots, x_n)=[f(x_1^*, \cdots, x_n^*)]^*$, we get a "dual system" A^*, isomorphic to A under the correspondence $f\to f^*$, $x\to x^*$. This principle of universal algebra contains the well-known duality principle of Boolean rings as a special case: $a^*=1-a$, and the dual of ab is $a\cup b=a+b-ab$. From this new point of view it is obvious that a similar duality exists in any ring with unity 1. It is also shown that the idempotent elements of any commutative ring form a Boolean ring. *G. Birkhoff* (Cambridge, Mass.).

Wade, L. I. Post algebras and rings. Duke Math. J. **12**, 389–395 (1945). [MF 12610]

Let R be a commutative ring with the following properties: (i) there exists an integer m such that $mx=0$ for all x in R; (ii) if $x\epsilon R$ and p is any prime divisor of m, there exists a y in R such that $x^p-x=py$; (iii) if $px=0$ and p^n is the maximum power of p dividing m, there is a y such that $x=p^{n-1}y$. The author shows that R is isomorphic to a subring of a direct sum of rings M_m, where M_m is the ring of integers modulo m. The proof is based on results of G. Birkhoff [Bull. Amer. Math. Soc. **50**, 764–768 (1944); these Rev. **6**, 33] and the reviewer [see the following review]. Post algebras, as defined by Rosenbloom [Amer. J. Math. **64**, 167–188 (1942); these Rev. **3**, 262], lead naturally to a consideration of rings of the type just described and the author uses this fact to obtain a representation theorem for Post algebras. Finally, there is given a direct proof that an

m-valued Post algebra P_m is a subdirect union of Post algebras $P_m(m)$, that is, Post algebras consisting only of a chain of m elements. *N. H. McCoy.*

McCoy, Neal H. Subdirectly irreducible commutative rings. Duke Math. J. **12**, 381–387 (1945). [MF 12609]

The author obtains necessary and sufficient conditions for a commutative ring R to be subdirectly irreducible, as defined by G. Birkhoff [Bull. Amer. Math. Soc. **50**, 764–768 (1944); these Rev. **6**, 33]. The conditions refer to properties of the unique minimal ideal and the divisors of zero. There are two cases, according as R consists entirely of divisors of zero or does not. In the latter case it is further shown that all divisors of zero are nilpotent if R has the descending chain condition on ideals. *I. Kaplansky.*

Kaplansky, Irving. A note on groups without isomorphic subgroups. Bull. Amer. Math. Soc. **51**, 529–530 (1945). [MF 12810]

Let R be a commutative principal ideal ring each of whose proper residue class rings is finite. Let V be a vector space over R such that (1) for every $v\epsilon V$ there is an $\alpha\neq 0$ in R with $\alpha v=0$; (2) there exists an integer r such that every finite subset of V can be spanned by r elements of V. It is proved that V is not isomorphic to any proper subspace. When R is the ring of integers the theorem was proved by R. A. Beaumont [same Bull. **51**, 381–387 (1945); these Rev. **7**, 5]. *I. S. Cohen* (Cambridge, Mass.).

Everett, C. J. The basis theorem for vector spaces over rings. Bull. Amer. Math. Soc. **51**, 531–532 (1945). [MF 12811]

Let $M=u_1K+\cdots+u_mK$ be a vector space of m basis elements over a ring K with unity. A necessary and sufficient condition that every subspace of M has a basis of at most m elements is that K is a right principal ideal ring without zero-divisors. *I. S. Cohen* (Cambridge, Mass.).

Levitzki, Jakob. Solution of a problem of G. Koethe. Amer. J. Math. **67**, 437–442 (1945). [MF 12924]

The problem in question, raised by Koethe and more recently by Baer, is whether the maximal condition on the right ideals of a ring R insures the nilpotence of nil-ideals in R. The answer is affirmative, the following stronger result being proved. Let N be the radical of R (the union of the semi-nilpotent ideals as defined by the author), and assume the maximal condition for the right ideals containing N and contained in N; then all nil-ideals (left or right) are nilpotent. Assuming the maximal condition for both left and right ideals, the author shows that any nil-subring is nilpotent. Related results are also established.

I. Kaplansky (Chicago, Ill.).

Seidenberg, A. Valuation ideals in polynomial rings. Trans. Amer. Math. Soc. **57**, 387–425 (1945). [MF 12562]

Let $\mathfrak{O}=K[x, y]$ be a polynomial ring in two variables over an algebraically closed field K of arbitrary characteristic, and let v be a 0-dimensional valuation of $\mathfrak{O}$ such that $v(x)>0$, $v(y)>0$. A v-ideal in $\mathfrak{O}$ is an ideal which contains, with every one of its elements, all elements of equal or greater value. These ideals were first studied by Zariski [Amer. J. Math. **60**, 151–204 (1938)]; in the present paper the investigation is carried through by a direct and explicit method. The 0-dimensional v-ideals, when ordered by inclusion, form a simple sequence $\{\mathfrak{q}_i\}$, and consequently, if γ_i is the minimum of the values of the elements of $\mathfrak{q}_i$, then $\{\gamma_i\}$ is an increasing sequence of elements of the value group of v. Let D_i be the subgroup generated by $v(x)$, $\gamma_1, \cdots, \gamma_i$. An analysis of the v-ideals is given by means of the D_i and in particular by means of certain polynomials $f_j(x, y)$ which are explicitly constructed in terms of the D_i and whose values are among the γ_i. For example, every $\mathfrak{q}_i$ has a basis consisting of monomials in the f_j. In this way the study of a general valuation is reduced to that of an "irrational" valuation (that is, one whose value group consists of real, but not exclusively of rational, numbers) by means of the following fundamental theorem. For every positive integer m there exists an irrational valuation of $\mathfrak{O}$ whose first m v-ideals are $\mathfrak{q}_1, \cdots, \mathfrak{q}_m$. If v is irrational an explicit criterion is given for a v-ideal to be simple (that is, not the product of other ideals). Namely, if $\tau=\gamma_k$ is the first irrational γ_i, then, for $i\geqq k$, $\mathfrak{q}_i$ is simple if and only if one of the numbers γ_i, γ_{i+1} is rational and the other an integral multiple of τ; in addition, if $\mathfrak{q}_i$ is not simple, a factorization is given. Every v-ideal is subject to this criterion. The proof makes use of the continued fraction expansion of a certain multiple of τ. Finally, it is shown that under a quadratic transformation a simple v-ideal is sent into a simple v-ideal.

I. S. Cohen (Cambridge, Mass.).

Jacobson, N. The radical and semi-simplicity for arbitrary rings. Amer. J. Math. **67**, 300–320 (1945). [MF 12435]

Let z be an element of a ring $\mathfrak{A}$; an element z' of $\mathfrak{A}$ is called a right quasi-inverse of z if $z+z'+zz'=0$ (if $\mathfrak{A}$ has a unit element, this means that $(1+z)(1+z')=1$). If z has a quasi-inverse, it is called quasi-regular. A right ideal is called quasi-regular if its elements are all quasi-regular. The author proposes to define the radical of $\mathfrak{A}$ to be the join of all quasi-regular right ideals and justifies his definition by showing that many theorems of the theory of rings with minimal condition can be generalized in a natural way in terms of this notion of radical. Thus: (1) the radical is a two-sided ideal and can also be defined to be the join of the left quasi-regular ideals; (2) a ring whose radical consists of 0 only being called semi-simple, if $\mathfrak{R}$ is the radical of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{R}$ is semi-simple; (3) if $\mathfrak{A}$ has a unit element, its radical is the intersection of the maximal right ideals; (4) $\mathfrak{A}$ being called primitive if it has a maximal ideal $\mathfrak{J}$ such that $\mathfrak{A}:\mathfrak{J}=\{0\}$, a primitive ring is semi-simple and every semi-simple ring contains a family of two-sided ideals $\mathfrak{B}$ such that each $\mathfrak{A}/\mathfrak{B}$ is primitive and $\prod\mathfrak{B}=0$; (5) any primitive ring can be represented as a dense ring of linear transformations in a vector space over a division ring [cf. Jacobson, Trans. Amer. Math. Soc. **57**, 228–245 (1945); these Rev. **6**, 200]. It goes without saying that the new notions of radical and of semi-simple ring reduce to the classical ones in the case of rings which satisfy the minimal condition, while the notion of a primitive ring becomes that of a simple ring. In the case of a complete normed ring, it is shown that the radical consists of all elements z such that $\lim_{n\to\infty}(za)^n=0$ for every element a of the ring. In the case where the normed ring is commutative, the radical is also the set of elements z such that $\lim_{n\to\infty}\|z^n\|^{1/n}=0$, which shows that the Jacobson definition agrees in this case with the Gelfand definition [cf. I. Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941); these Rev. **3**, 51].

C. Chevalley (Princeton, N. J.).

Dubreil, Paul. L'indépendance linéaire dans un module sur un anneau non nécessairement commutatif. Bull. Sci. Math. (2) **67**, 84–100 (1943). [MF 12632]

Linear independence of elements of a module $\mathfrak{M}$ over a ring $\mathfrak{o}$ is discussed from the point of view of the general

theory of dependence [cf. T. Nakasawa, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 2, 235–255 (1935); 3, 45–69, 123–136 (1936); O. Haupt, G. Nöbeling and C. Pauc, J. Reine Angew. Math. 181, 193–217 (1940); these Rev. 1, 169]. It is shown that the axiom of exchange (if $(x_1, \cdots, x_r)$ are independent, but $(x_1, \cdots, x_r, y)$ and $(x_1, \cdots, x_r, z)$ are dependent, then $(x_1, \cdots, x_r, y, z)$ are dependent) holds if and only if $\mathfrak{o}$ is left regular (that is, $\mathfrak{o}$ has no zero divisor and any two elements in $\mathfrak{o}$ have a common left-multiple not equal to 0). It is shown that, if $\mathfrak{o}$ does not satisfy this condition and if the condition $\alpha a=0$ (α in $\mathfrak{o}$, a in $\mathfrak{M}$) implies either $\alpha=0$ or $a=0$, then it is possible to construct an arbitrarily large number of elements of $\mathfrak{M}$ which are linearly independent over $\mathfrak{o}$. *C. Chevalley* (Princeton, N. J.).

Dieudonné, Jean. Les déterminants sur un corps non commutatif. Bull. Soc. Math. France 71, 27–45 (1943). [MF 13231]

Let K be an arbitrary division ring, and M_n the multiplicative group of all regular matrices of order n with elements in K. If C_n is the commutator subgroup of M_n, the author shows that, for each positive integer n, $M_n/C_n \cong M_1/C_1$, where C_1 is naturally the commutator subgroup of the multiplicative group M_1 of K itself. The proof, which uses induction on n, consists in exhibiting a well-defined homomorphism $X \rightarrow \Delta_n(X)$ of M_n on the group M_1/C_1, with kernel C_n. If K happens to be commutative, this $\Delta_n(X)$ coincides with the usual determinant of X and, in any case, it is defined to be the determinant of X and shown to have a number of familiar properties of determinants. Cramer's rule can be extended to the noncommutative case, with due attention to the fact that $\Delta_n(X)$ is not an element of K but an element of the quotient group M_1/C_1. There are a number of other results, of which we mention only the following. If $n>2$, every invariant subgroup of M_n, not contained in the center of M_n, contains the commutator subgroup C_n. This is also true for $n=2$ if the center of K has more than three elements. In general, the paper follows the methods of Jordan, Burnside and Dickson [L. E. Dickson, Linear Groups, Teubner, Leipzig, 1901, pp. 75–88] which were effective in the commutative case. *N. H. McCoy.*

Levit, Robert J. Fields in terms of a single operation. Trans. Amer. Math. Soc. 57, 426–440 (1945). [MF 12563]

N. Wiener first stated a system of postulates for a field in terms of a single binary operation ∇, where $a\nabla b$ behaves formally like $1-a/b$ [same Trans. 21, 237–246 (1920)]. The present author points out that, while the field is not closed under ∇, its left inverse Δ, where $a\Delta b$ behaves formally like $a(1-b)$, is class closing; he gives a system of five postulates for a field in terms of Δ. A new system of six postulates is also given for a field in terms of Wiener's ∇, and both systems of postulates are proved necessary, sufficient and independent. The concept of equivalence of two sets of operations in a mathematical system, relative to given postulates, is introduced by defining the sets to be equivalent if they both satisfy the postulates. In particular, therefore, a theorem deducible from a given set of postulates in terms of one set of operations will remain true if these operations are replaced throughout by the operations of an equivalent set. The following interesting result is obtained in this connection. The most general pair of rational binary operations $(\oplus, \odot)$ which are equivalent to the ordinary sum and product in every field are of the form $a\oplus b=a+b-Z$; $a\odot b=(ab-Z(a+b)+UZ)/(U-Z)$, where Z and U are arbitrary distinct field elements which play the role of zero and unit elements for the operations $(\oplus, \odot)$. The problem of determining necessary and sufficient conditions to ensure that a given binary operation define a field is proposed. *S. A. Jennings* (Vancouver, B. C.).

Foster, Alfred L., and Bernstein, B. A. A dual-symmetric definition of field. Amer. J. Math. 67, 329–349 (1945). [MF 12915]

Continuing their researches on duality in commutative rings [Duke Math. J. 11, 603–616 (1944); these Rev. 6, 34] the authors develop two self-dual sets of postulates for a field in terms of products ab and dual symmetric differences $a\Delta b=a+b-ab$. They postulate that, under ab and $a\Delta b$, the elements, except 0 and 1, respectively, form a group. They write a^{-1} as a', the inverse $a/(a-1)$ of a under Δ as a^0, and $a^{0'0}=1-a$ as a^*. They then postulate that $ab=a\Delta(a^0b^*)$, dually that $a\Delta b=a(a'\Delta b^*)$, and that $(a^0a'a^*)=(b'\Delta b^0\Delta b^*)'$. The independence of the postulates is not discussed. *G. Birkhoff* (Cambridge, Mass.).

Kaplansky, Irving. Maximal fields with valuations. II. Duke Math. J. 12, 243–248 (1945). [MF 12595]

[Part I appeared in the same J. 9, 303–321 (1942); these Rev. 3, 264.] Let K be a field maximal in a valuation with value group Γ and residue class field $\mathfrak{K}$ of the same characteristic as K. It is shown that if Γ is discrete then K is the power series field over $\mathfrak{K}$ with exponents in Γ; Γ is defined to be discrete if its isolated subgroups $\{\Delta_\rho\}$ are well-ordered with respect to inclusion and if each $\Delta_{\rho+1}/\Delta_\rho$ is the additive group of integers. Together with results of part I this implies the following: K is always a power series field if $\mathfrak{K}$ is of characteristic 0 or Γ is discrete. If $\mathfrak{K}$ is of characteristic p and Γ is nondiscrete, then K is a power series field provided that $\Gamma=p\Gamma$ and every polynomial $\sum a_i x^{p^i}$ ($a_i\epsilon\mathfrak{K}$) has a root in $\mathfrak{K}$; neither of these two hypotheses may be omitted. *I. S. Cohen* (Cambridge, Mass.).

Pétresco, Julien. Sur le problème inverse à la théorie de Galois dans les corps finis. Bull. Math. Soc. Roumaine Sci. 45, 113–123 (1943). [MF 12762]

For Galois fields (fields with a finite number of elements) the problem of the existence of extension fields with given Galois group is already solved: every Galois field has one and only one extension field of given degree, and this field is normal with cyclic Galois group. The author presents proofs of these facts, which are well known. He also discusses the equation satisfied by an arbitrary element of a normal extension field, making use of parametric formulas for the coefficients and deriving a criterion for deciding whether two equations define the same extension field. [A much simpler criterion is given by the fact that, for Galois fields, two irreducible equations of the same degree define the same extension field.] *G. Whaples.*

Tuan, Hsio-Fu. A note on the replicas of nilpotent matrices. Bull. Amer. Math. Soc. 51, 305–312 (1945). [MF 12273]

The concept of a replica of a matrix was introduced by Chevalley [Amer. J. Math. 65, 521–531 (1943); these Rev. 5, 171], who proved that, if the underlying field K is of characteristic 0, the replicas of a nilpotent matrix Z are its scalar multiples. The author gives an elementary proof of this theorem and shows in addition that, if K is of characteristic p, the replicas of Z are those and only those matrices of the form $\sum t_i Z^{p^i}$, $t_i\epsilon K$. *I. S. Cohen.*

Chevalley, Claude, and Tuan, Hsio-Fu. On algebraic Lie algebras. Proc. Nat. Acad. Sci. U. S. A. **31**, 195–196 (1945). [MF 12797]

In a previous paper [Amer. J. Math. **65**, 521–531 (1943); these Rev. **5**, 171] Chevalley defined a replica of a given matrix X to be any matrix Y such that Y admits as its invariants all the tensor invariants of X. The authors define a Lie algebra of matrices over a field K of characteristic 0 to be algebraic if the algebra contains every replica of each of its matrices. Several consequences of this definition are announced. For example, the Lie algebra of derivations of any algebra (associative or not) over K is algebraic, as is the derived algebra of any matric Lie algebra over K; more generally, any Lie algebra whose radical consists only of nilpotent matrices has this property. In particular, algebraic Lie algebras over the complex field are completely characterized as the Lie algebras of algebraic groups of matrices with complex coefficients, an algebraic group of matrices being one for which the condition that a general matrix belongs to the group may be expressed in the form of a system of algebraic equations on the coefficients of the matrix.

S. A. Jennings (Vancouver, B. C.).

Gomes, Ruy Luís. Example of algebras which admit a particular type of involution. Gaz. Mat., Lisboa **6**, no. 23, 1–3 (1945). (Portuguese) [MF 12531]

An element a is said to be symmetric (or antisymmetric) with respect to an involution J if $Ja=a$ (or $-a$). The particular type of involution discussed is one for which all symmetric elements are of the form αu, where α is an element of the field over which the algebra is constructed and u is the unit. *G. Y. Rainich* (Ann Arbor, Mich.).

de Mira Fernandes, A. Algebras in involution. Gaz. Mat., Lisboa **6**, no. 24, 1 (1945). (Portuguese) [MF 12958]

After giving a simpler discussion of the case treated by Gomes in the note reviewed above, the author shows, for an algebra with an involution, that it is always possible to choose a basis in which every element is either symmetric or antisymmetric, and that the numbers of these elements are invariants. *G. Y. Rainich* (Ann Arbor, Mich.).

Etherington, I. M. H. Transposed algebras. Proc. Edinburgh Math. Soc. (2) **7**, 104–121 (1945). [MF 12354]

A linear algebra is defined by a multiplication table $u_iu_j=\sum\gamma_{ijk}u_k$. The transposes of an algebra are then defined by $u_iu_j=\sum\delta_{ijk}u_k$, where $\delta_{ijk}=\gamma_{ijk}, \gamma_{ikj}, \gamma_{kji}, \gamma_{jik}, \gamma_{jki}$, or γ_{kij}. The author considers the question of whether an algebra and its transposes are isotopic and studies the transposition of direct sums and direct products. Transposition is applied to complex numbers, quaternions and Abelian group algebras and the resulting algebras are shown to be palintropic, that is, to have the property $(x^a)^b=(x^b)^a$. Other palintropic algebras are considered. The direct algebra of an algebra A is defined to be the algebra of polynomials in A under direct sum and direct product and properties of the direct algebra of the algebra of complex numbers are derived.

A. A. Albert (Chicago, Ill.).

Svartholm, N. On the algebras of relativistic quantum theories. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **12**, no. 9, 94–108 (1942). [MF 12588]

One of the approaches toward the establishment of the relativistic wave equations for the elementary particle in quantum theory is that of considering a set of N operators ξ_m such that

$$[\xi_m[\xi_j, \xi_k]]=\delta_{jm}\xi_k-\delta_{km}\xi_j.$$

Here $[\xi_j, \xi_k]=\xi_j\xi_k-\xi_k\xi_j$ and δ_{jm} is the Kronecker delta. These operations generate an algebra over the field of all complex numbers if we assume that there exist complex numbers a_j such that $\xi_m^{s+1}=a_s\xi_m^s+\cdots+a_0$. The author considers the cases $s=1, 2$. In the case $s=1$ the algebra is a Clifford algebra and so is a total matric algebra or a direct sum of two such algebras. In the case $s=2$ the algebra is also shown to be a direct sum of total matric algebras.

A. A. Albert (Chicago, Ill.).

Schafer, R. D. On a construction for division algebras of order 16. Bull. Amer. Math. Soc. **51**, 532–534 (1945). [MF 12812]

The author considers algebras A of order sixteen defined, in terms of the well-known alternative Cayley-Dickson division algebra of order eight, by

$$(a+vb)(x+vy)=ax+g(y\bar{b})+v(\bar{a}y+xb).$$

Here a, b, x, y range over all Cayley numbers and the defining parameter g is any Cayley number. The author shows that, if the reference field F is the field of all real numbers, then A is never a division algebra. However, if F is such that there exist Cayley numbers g whose norms are not the squares of elements of F, then the corresponding algebras are division algebras. *A. A. Albert* (Chicago, Ill.).

THEORY OF GROUPS

Everett, C. J., and Ulam, S. On ordered groups. Trans. Amer. Math. Soc. **57**, 208–216 (1945). [MF 12130]

An o-group is a partially ordered set G which is "directed" (that is, given a and b in G, c exists such that $c\geqq a$ and $c\geqq b$), a group, and "homogeneous" (that is, $x\geqq y$ implies $a+x+b\geqq a+y+b$ for all a, b). If G is a lattice under $\geqq$, it is called an l-group. The authors show that an o-group G can be completed (actually, by cuts) if and only if it is "integrally closed" (that is, $na\leqq b$ for all positive integers n implies $a\leqq 0$); this was proved in case $x+y=y+x$ by A. H. Clifford. In other o-groups, limit elements of "o-regular" (generalized Cauchy) sequences can be adjoined. Also, if G is integrally closed and if its commutator-subgroup is in the center, then G is commutative. By considering the l-group of continuous monotone increasing functions of [0, 1] onto itself under substitution, and the o-subgroup of algebraic functions of this kind, it is shown that (i) in l-groups, the set of $n(a+b-a-b)$ is unbounded for some a, b; (ii) in l-groups, we can have $na>nb$, where $n, a, b>0$, without $a>b$; (iii) $na\leqq b$ for all integers n may imply $a=0$ in an o-group which is not integrally closed (this is impossible in l-groups). The preceding results settle various questions raised by the reviewer. The free group F_2 with two generators is made into an o-group, but not into an l-group. [A. Tarski and the reviewer, in independent unpublished work, made F_n into a simply ordered l-group.]

G. Birkhoff (Cambridge, Mass.).

Morgado, José. Modern algebra, 1. Groups. Cadernos de Análise Geral, no. 3. Junta de Investigação Matemática, Pôrto, 1945. 31 pp. (Portuguese) [MF 12907]

This pamphlet deals with equivalence classes, groups (defined postulationally), groups of transformations, subgroups, lattices and modular lattices. *G. Y. Rainich.*

Bose, R. C., and Chowla, S. On a method of constructing a cyclic subgroup of order $p+1$ of the group of linear fractional transformation mod p. Science and Culture 10, 558 (1945) = Proc. Lahore Philos. Soc. 7, 53 (1944). [MF 12838]

If a, b, c, d are marks of $GF(p)$, $ad-bc\neq 0$, the transformations $x^*=(ax+b)/(cx+d)$ form a group of order $p(p^2-1)$. The authors state without proof that a cyclic subgroup of order $p+1$ is formed by the transformations $x^*=a_2/(a_1+x)$, where $x^2=a_1x+a_2$ is the equation, irreducible in $GF(p)$, satisfied by a primitive element of $GF(p^2)$; and that the generalization from p to p^n is easy. [The reviewer suggests a short proof. In homogeneous coordinates the equation

$$x^k(1, x)=(1, x)\begin{pmatrix}0 & a_2\\ 1 & a_1\end{pmatrix}^k$$

is assumed for $k=1$ and can be proved by induction. Since x is primitive in $GF(p^2)$, the smallest value of k for which the last factor is a diagonal matrix is $p+1$. Furthermore, $x^{p+1}=-a_2$, $x+x^p=a_1$.] *J. S. Frame.*

Venkatarayudu, T. Immanants of a matrix associated with a finite Abelian group. Proc. Indian Acad. Sci., Sect. A. 21, 103–104 (1945). [MF 12614]

If $a_{ij}=S_iS_j$, where $S_1, S_2, \cdots, S_N$ are the elements of an Abelian group of order N, then all the immanents of the matrix $[a_{ij}]$ vanish except the permanent, which is equal to $N!E$. *G. de B. Robinson* (Toronto, Ont.).

Miller, G. A. Groups having a small number of sets of conjugate subgroups. Proc. Nat. Acad. Sci. U. S. A. 31, 147–150 (1945). [MF 12447]

The author determines those groups having not more than six sets of conjugate subgroups. [Cf. his former paper [same Proc. 30, 359–362 (1944); these Rev. 6, 145], where a similar limitation is imposed upon the number of sets of conjugate operators.] *G. de B. Robinson* (Toronto, Ont.).

Frucht, Roberto. The subgroups of the complete monomial groups of degree 2. Univ. Nac. Tucumán. Revista A. 4, 47–54 (1944). (Spanish) [MF 13018]

Let H be a finite permutation group of order h on the letters $x_1, x_2, \cdots, x_n$ and let H' be the same group written on the letters $y_1, y_2, \cdots, y_n$. Let T be the permutation $(x_1y_1)(x_2y_2)\cdots(x_ny_n)$. The group considered is that on $2n$ letters generated by H, H' and T. It is clear that this group can be written in the form $H\times H'+(H\times H')T$, where $H\times H'$ is the direct product of H and H'; it is easily seen to be identical with the complete monomial group of degree 2 in the sense of Ore [Trans. Amer. Math. Soc. 51, 15–64 (1942); these Rev. 3, 197]. The author determines all subgroups of this group in terms of the various subgroups of H.

H. W. Brinkmann (Swarthmore, Pa.).

Dubuque, P. E. Sur les sous-groupes d'ordre fini dans un groupe infini. Rec. Math. [Mat. Sbornik] N.S. 10(52), 147–150 (1942). (Russian. French summary) [MF 12829]

The following theorem is proved. Let $\mathfrak{A}$ be a subgroup of $\mathfrak{G}$, and let $\mathfrak{H}$ be a group which can be obtained from $\mathfrak{A}$ by adjoining to it a finite number of elements $P_1, \cdots, P_n$ of $\mathfrak{G}$. If each of the elements $P_1, \cdots, P_n$ is of finite order, and if each element of $\mathfrak{H}$ which is the conjugate of some element P_i belongs to one of the cosets $\mathfrak{A}, \mathfrak{A}P_1, \cdots, \mathfrak{A}P_n$, then $\mathfrak{A}$ is a subgroup of $\mathfrak{H}$ of finite index. This theorem contains as a particular case a result previously obtained by W. K. Turkin and the author [C. R. Acad. Sci. Paris 205, 435–437 (1937); same Rec. N.S. 3(45), 425–429 (1938)] and also yields a result of A. P. Dietzmann [C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 71–76 (1937)]. *H. P. Thielman.*

Prokofieff, A. N. Einige Eigenschaften der Zerlegungen einer Gruppe nach ihren Untergruppen. Rec. Math. [Mat. Sbornik] N.S. 10(52), 143–145 (1942). (Russian. German summary) [MF 12828]

This paper generalizes some of the results obtained by A. A. Kulakoff [same Rec. N.S. 2(44), 357–359, 1003–1006 (1937); 3(45), 187–189 (1938); 4(46), 371–373 (1938)]. It deals with the distribution of the elements of a group within two cosets taken with respect to different subgroups of the given group. A typical result is as follows. Let $\mathfrak{A}$ and $\mathfrak{B}$ be subgroups of $\mathfrak{G}$, and let M and N be elements of $\mathfrak{G}$. If the cosets $M\mathfrak{A}$ and $\mathfrak{B}N$ have an element in common, then the total number of elements common to these two cosets is equal to the order of the logical product of the subgroups $\mathfrak{A}$ with each of the subgroups $N^{-1}\mathfrak{B}N$ and $M^{-1}\mathfrak{B}M$.

H. P. Thielman (Ames, Iowa).

Krutik, B. A. Über einige Eigenschaften der endlichen Gruppen. Rec. Math. [Mat. Sbornik] N.S. 10(52), 239–247 (1942). (Russian. German summary) [MF 12833]

The following theorems are proved. (1) Let A be an element of a subset $\mathfrak{S}$ of a group $\mathfrak{G}$, and let $\mathfrak{N}_A^{\mathfrak{S}}$ be the set consisting of all elements X of $\mathfrak{G}$ which satisfy the condition $XAX^{-1}\subset\mathfrak{S}$. Let the class (A) of $\mathfrak{G}$ be the set of all conjugates of A relative to $\mathfrak{G}$. Then if $\mathfrak{N}_A^{\mathfrak{S}}$ is a group, the number of different elements of $\mathfrak{S}$ which belong to the class (A) is a divisor of the order of the class (A). (2) Let the element A of $\mathfrak{G}$ be of order m and let d be a divisor of m. The number of different elements of the form A^{s+1}, $s\equiv 0 (\mathrm{mod}\ d)$, which belong to the class (A) of $\mathfrak{G}$ is a divisor of $\varphi(m)/\varphi(d)$, where $\varphi(k)$ is the Euler φ-function. (3) If the order of the class (A) of $\mathfrak{G}$ and $\varphi(m)/\varphi(d)$ are relatively prime to each other, where m is the order of A, then the element A is not a conjugate of A^{s+1} relative to $\mathfrak{G}$ if $s\neq 0$ and $s\equiv 0$ (mod d). (4) Let the element A of $\mathfrak{G}$ be of order $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$, where each p is a prime number and $p_1<p_2<\cdots$. Let $d=p_1^{\beta_1}p_2^{\beta_2}\cdots p_t^{\beta_t}$ $(\beta_i\neq 0)$ be a divisor of m, and let $(k, m)=1$. Then if A is a conjugate of A^{kd+1} relative to $\mathfrak{G}$, A is a conjugate of each of the $\varphi(m)/\varphi(d)$ elements of order m which have the form A^{s+1}, $s\equiv 0$ (mod d). The only exception can occur when $p_1=2$ while $\beta_1=1$. The author points out that earlier results of Burnside, Frobenius, Tchounikhin, and Turkin can be obtained as direct consequences of these theorems. *H. P. Thielman* (Ames, Iowa).

Beaumont, Ross A. Groups with isomorphic proper subgroups. Bull. Amer. Math. Soc. 51, 381–387 (1945). [MF 12515]

Groups which are not isomorphic to any proper subgroup have been discussed by R. Baer [same Bull. 50, 267–278 (1944); these Rev. 5, 228]. In this paper we have a characterization of those completely reducible groups which have isomorphic proper subgroups. A group is said to have finite rank r if every finite set of elements is contained in a subgroup with at most r generators; completely reducible

groups are those which are direct products of groups of rank one. The author shows that a completely reducible group has an isomorphic proper subgroup if and only if it has a direct factor isomorphic to a group of one of the following three types: (i) primary Abelian groups of infinite rank; (ii) Abelian groups, which do not admit the rational numbers as operators, having all their elements other than the identity of infinite order; (iii) vector spaces of infinite dimension over the rational field. *S. A. Jennings.*

Bergström, Harald. Über Erweiterungen abelscher Gruppen. Ark. Mat. Astr. Fys. **28B**, no. 13, 7 pp. (1942). [MF 12326]

In this paper the author proposes to study the special Hölder problem of finding all extensions G of a given Abelian group A by a given group F. In this connection the so-called factor system $c(\sigma, \tau)$ is known to play a fundamental role, $c(\sigma, \tau)\varepsilon A$ being defined for every pair of elements $\sigma, \tau\varepsilon F$. The author is interested in the possible choice of an associated factor system $c'(\sigma, \tau)$ under suitable assumptions. His investigation reduces to the study of the special case for which F contains a cyclic normal subgroup $\{\mu\}$ of finite order. His fundamental lemma asserts that in this case an associated factor system $c'(\sigma, \tau)$ can always be chosen so that all $c'(\sigma, \tau)$ belong to the subgroup C_μ of all those elements c satisfying $c^\mu=c$ of the subgroup C of A generated by all $c(\sigma, \tau)$, where we write $V(\mu)aV(\mu)^{-1}=a^\mu$ for $a\varepsilon A$, $V(\mu)$ being the chosen representative of μ. All his theorems depend on this lemma. In the course of its proof, writing $\sigma\mu\sigma^{-1}=\mu^{k_\sigma}$ for $\sigma\varepsilon F$, the author states: "Ersichtlich ändert der Automorphismus $a\rightarrow a^{\mu^{k_\sigma}}$ alle Nebenklassen von C nach C_μ. Daher definiert $X\rightarrow X^{1-\mu^{k_\sigma}}$ einen Automorphismus von C/C_μ, den ich mit $\bar\mu_{k_\sigma}$ bezeichne." However, it seems to the reviewer that this statement is not true in general, as is shown by the following example. Let G be a group of order p^4 for an odd prime p, defined by the relations $E_i^p=1$ $(i=1, 2, 3, 4)$, $(E_3, E_4)=E_3^{-1}E_4^{-1}E_3E_4=E_2$, $(E_2, E_4)=E_1$. Then $A=\{E_1, E_2\}$ is the Abelian commutator subgroup of G and $Z=\{E_1\}$ is the center of G. Clearly $G/A\cong F=\{\sigma, \mu\}$ with $\sigma^p=\mu^p=(\sigma, \mu)=1$; here $k_\sigma=1$. Putting $V(\mu)=E_4$, $V(\sigma)=E_3$, $V(\sigma\mu)=V(\mu\sigma)=E_1^{-1}E_4E_3=E_2^{-1}E_3E_4$, we find $c(\mu, \sigma)=E_1$, $c(\sigma, \mu)=E_2$ and $C=A$. Furthermore, $C_\mu=Z$ and $(E_1^xE_2^y)^{1-\mu}=E_1^y$. It follows that $\bar\mu_{k_\sigma}=\bar\mu_1$ is the trivial homomorphism of C/C_μ onto C_μ/C_μ. This contradicts the author's statement. That the lemma itself is also negated thereby can be seen from a simple calculation for which we put $V'(\mu)=E_1^xE_2^yE_4$, $V'(\sigma)=E_1^zE_2^wE_3$ and find

$$V'(\sigma)V'(\mu)V'(\sigma)^{-1}V'(\mu)^{-1}=c'(\sigma, \mu)c'(\mu, \sigma)^{-1}=E_2E_1^{w-1}\notin C_\mu,$$

whence not both $c'(\sigma, \mu)$ and $c'(\mu, \sigma)$ can be elements of C_μ.
H. F. Tuan (Princeton, N. J.).

Bergström, Harald. Vereinfachter Beweis des Hauptidealsatzes der Klassenkörpertheorie. Ark. Mat. Astr. Fys. **29B**, no. 6, 6 pp. (1943). [MF 12016]

It is well known that Artin first observed the close connection between the Hauptidealsatz of class field theory and a certain theorem on metabelian groups (groups with Abelian commutator subgroups). The first proof of this group-theoretical result, due to Furtwängler, is quite lengthy. The author proposes to give a simplified proof along Furtwängler's lines. His procedure is based on the fundamental lemma of the paper reviewed above, which appears to be incorrect. Theorem 4 of the present paper asserts that the commutator subgroup A of a metabelian group G belongs to the center of G if and only if the order of A contains only prime factors which enter into the order of G/A. From this result, it would follow that every metabelian p-group has its commutator subgroup contained in its center, which is not true; compare the counterexample in the preceding review. [It should be emphasized that many mathematicians use "metabelian group" for a group whose commutator subgroup lies in its center.] *H. F. Tuan.*

Bergström, Harald. Struktur der Erweiterungen abelscher Gruppen. Ark. Mat. Astr. Fys. **30A**, no. 4, 10 pp. (1944). [MF 12004]

The author states the following theorem. "Let $\mathfrak{B}$ be the direct product of l cyclic groups $\mathfrak{P}_i$ of order n; let Γ be a group which has a subgroup Γ_0 of index l; let Ω be a group which has a cyclic normal subgroup $\mathfrak{Z}$ of order n such that $\Gamma_0\cong\Omega/\mathfrak{Z}$. Then there exists a group $\mathfrak{G}$ which is an extension of $\mathfrak{B}$ by Γ and which satisfies the following conditions: $\mathfrak{N}_1$ being the normalizator of $\mathfrak{P}_1$ in $\mathfrak{G}$, there is an isomorphism of $\mathfrak{G}/\mathfrak{B}$ with Γ which maps $\mathfrak{N}_1/\mathfrak{B}$ onto Γ_0 and there is an isomorphism of $\mathfrak{N}_1/\mathfrak{P}_2\times\cdots\times\mathfrak{P}_l$ onto Ω which maps $\mathfrak{B}/\mathfrak{P}_2\times\cdots\times\mathfrak{P}_l$ onto $\mathfrak{Z}$. Moreover, $\mathfrak{G}$ is uniquely determined by these conditions except for isomorphism." This last assertion is false as can be seen from the following example: $n=2$, $l=3$, $\mathfrak{P}_i=\{E, A_i\}$ (where E is the unit element), $\Gamma=\{\epsilon, \sigma, \sigma^2\}$ (where ϵ is the unit element), $\Gamma_0=\{\epsilon\}$, $\Omega=\mathfrak{Z}=\mathfrak{P}_1$, and either $\sigma A_1\sigma^{-1}=A_2$, $\sigma A_2\sigma^{-1}=A_3$ or $\sigma A_1\sigma^{-1}=A_1A_2$, $\sigma A_2\sigma^{-1}=A_1$, $\sigma A_3\sigma^{-1}=A_2$. The proof of the author seems to become correct if one adds the assumption that the operations of Γ permute the groups $\mathfrak{P}_i$ among themselves. However, it is not true, contrary to what the author seems to assume, that, Γ being a group of automorphisms of a finite Abelian group $\mathfrak{G}$, it is possible to represent $\mathfrak{G}$ as a direct product of cyclic groups which are permuted among themselves by the operations of Γ.
C. Chevalley (Princeton, N. J.).

Eriksson, H. Adolf S. Spinor representation of rotations and Dirac's equations in five-dimensional space. Ark. Mat. Astr. Fys. **29A**, no. 14, 9 pp. (1943). [MF 12014]

The spinor representation of the five-dimensional orthogonal group is given explicitly, in terms of the Cayley parameters of an orthogonal matrix. The Dirac equations are formulated for the five-dimensional space. Taking the four-dimensional orthogonal group as a subgroup of the five-dimensional group, Lorentz transformations may be considered as a special case. *R. Brauer* (Toronto, Ont.).

Littlewood, D. E. On invariant theory under restricted groups. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 387–417 (1944). [MF 11503]

[This is a sequel to a paper in the same Trans. Ser. A. **239**, 305–365 (1944); these Rev. **6**, 41.] Here the subject is the restricted linear groups, including the orthogonal and the symplectic group. The author discusses the problem raised by Klein in his Erlanger Programm, namely the relation between the concomitants of a set of ground forms F for a subgroup G of the full linear group Γ, and those of a certain extended set of ground forms for the full group Γ. These two concomitant systems are identical in the cases here discussed, but the complete answer to the questions raised has not yet been found [cf. Weitzenböck, Encyklopädie der Mathematischen Wissenschaften III E 1 (vol. III$_3$, no. 6), 1922, in particular, p. 19]. The tensor methods are used, as an alternative to previous symbolic methods; the requisite group characters and S-functions are elab-

orated, by algebraic methods, for the orthogonal and the symplectic groups. This means that a quadratic form and a skew symmetric bilinear form, respectively, are adjoined to the given ground forms, in order to discuss these particular subgroups.

Concomitants up to degree five for a single quadratic (in three, four, five or six variables) are completely found for the orthogonal group; also for the ternary cubic to degree four; for the quaternary line complex to degree five for the full group and for the symplectic group. There follow various applications to forms in independent systems of variables, which illustrate intransitive groups.

The method gives accurate and precise information step by step according to the degree; it confirms the results obtained by symbolic methods, and in some cases discloses errors. It is the logical outcome for higher forms of the method of generating functions developed by Cayley and Sylvester for binary forms. *H. W. Turnbull.*

Markoff, A. On the existence of periodic connected topological groups. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 8, 225–232 (1944). (Russian. English summary) [MF 12299]

In a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 299–301 (1941); these Rev. 3, 36] the author gave an example of a periodic group of arbitrary power which does not admit a connected topology. In the present paper he raises the question of whether there exist connected periodic topological groups. The answer is affirmative and is given by the following theorem. For every $\mathfrak{m}\geqq 2^{\aleph_0}$, there exists a connected topological group whose power is $\mathfrak{m}$ such that the square of any element is the identity. The point of departure for the construction of such a group is an arbitrary connected completely regular topological space X; the group constructed contains a subset homeomorphic to X. *M. S. Knebelman* (Pullman, Wash.).

Markoff, A. On free topological groups. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 3–64 (1945). (Russian. English summary) [MF 12770]

The underlying idea of this paper is to trace out the deep analogy between the theories of discrete groups and topological groups. It is found that many fundamental theorems may be proved if instead of normal subgroup one uses a closed normal subgroup; for factor group, a topological factor group; for homomorphism either a continuous or an open continuous homomorphism; for set, a completely regular space. The author states twenty definitions, proves twenty-nine theorems and formulates six unsolved problems. We shall give only a few of them. A subset X of a topological group G generates the group topologically if there exists no proper closed subgroup of G containing X. It is shown that if X is a completely regular space there exists a topological group F such that (1) X is a subspace of F, (2) X generates F topologically, (3) if φ is any mapping of X into any topological group G, there exists a continuous homomorphism Φ of the topological group F into G such that $\Phi x=\varphi x$ for every x of X. This topological group F is unique up to topological isomorphisms carrying every point of X into itself; it is called the free topological group of the space X. It is then proved that every completely regular space is closed in its free topological group and that every topological group is topologically isomorphic to a topological quotient group of a free topological group. Another important definition is as follows: N is a norm of G if N is a real function in a group G such that (1) $Nl(G)=0$ and (2) $N(xy^{-1})<N(x)+N(y)$ $(x\epsilon G,\ y\epsilon G)$, $l(G)$ being the unit element in G. If $\mathfrak{N}$ is a set of norms $N_1, N_2, \cdots$, it is called a multinorm in G if (1) $N_1+N_2\epsilon\mathfrak{N}$ for every $N_1\epsilon\mathfrak{N}$, $N_2\epsilon\mathfrak{N}$; (2) for any $N\epsilon\mathfrak{N}$ and $a\epsilon G$, $N_a(x)=N(a^{-1}xa)$ belongs to $\mathfrak{N}$; (3) for every $a\epsilon G$, $a\neq l(G)$, there exists a norm N in $\mathfrak{N}$ such that $N(a)\neq 0$. By means of this it is shown that a topological group is determined by a multinorm. The following are six conjectural theorems, some of which are known to be true for Abelian or countable groups. (I) The free topological groups of two completely regular spaces are topologically isomorphic if and only if these spaces are homeomorphic. (II) Same as (I) for free Abelian topological groups. (III) Every uncountable group admits a nonnormal topology. (IV) Every infinite group admits a nontrivial topology. (V) If every unconditionally closed proper subgroup of a group G is of index $2^{\aleph_0}$, then G admits a connected topology. (VI) Every unconditionally closed subset of a group G is an algebraic subset of this group.

M. S. Knebelman (Pullman, Wash.).

Baer, Reinhold. The homomorphism theorems for loops. Amer. J. Math. 67, 450–460 (1945). [MF 12926]

Proofs are given of the isomorphism theorems and the Jordan-Hölder-Schreier-Zassenhaus theorem for loops. The methods of proof are similar to those used by Zassenhaus [Lehrbuch der Gruppentheorie, vol. 1, Hamburger Math. Einzelschr. no. 21 (1937)]. The theorems themselves have been proved before, as the author points out. The distinction between normal and invariant subloops is forcefully brought out. *H. Campaigne* (Washington, D. C.).

ANALYSIS

Watson, G. N. A problem in elementary analysis proposed by Mordell. Proc. London Math. Soc. (2) 48, 391–400 (1945). [MF 12047]

The author proves a conjecture of Mordell [same Proc. (2) 48, 339–390 (1944); these Rev. 6, 257] which can be stated in the following form. Let

$$f(x, y)=(1+y)^x+(1-y)^x-y^x-1$$

$(0\leqq x\leqq 1;\ 0\leqq y\leqq 1)$, and let

$$F(x, y)=(2y+3)^x+(3y-2)^x-y^x-1$$

$(0\leqq x\leqq 1;\ \frac{2}{3}\leqq y\leqq 1)$. Then the curves $f(x, y)=0$ and $F(x, y)=0$ have one and only one point of intersection, (ξ, η), inside the rectangle whose sides are $x=0$, $x=1$, $y=\frac{2}{3}$, $y=1$. Moreover, if (ξ', η') is any point on the curve $f(x, y)=0$ and inside the rectangle, with $\xi'>\xi$, then $F(\xi', \eta')$ is positive.

D. C. Spencer (Stanford University, Calif.).

Uspensky, J. V. On the arithmetico-geometric means of Gauss. Math. Notae 5, 1–28 (1945). (Spanish) [MF 12617]

If a and b are two positive numbers, their arithmetico-geometric mean is $M(a, b)=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$, where $a_0=a$, $b_0=b$, $a_{n+1}=\frac{1}{2}(a_n+b_n)$, $b_{n+1}=(a_nb_n)^{\frac{1}{2}}$. The paper is the first part of an expository account dealing with Gauss's theory of $M(a, b)$ and its application to the theory of modular functions. In this part, the author derives the differential equation satisfied by $M(a, b)$; he also shows how this function can be used in the calculation of the complete elliptic integral of the second kind. *R. P. Boas, Jr.*

v. Sz. Nagy, Béla. Über Carlsonsche und verwandte Ungleichungen. Mat. Fiz. Lapok 48, 162–175 (1941). (Hungarian. German summary) [MF 12369]

An inequality of Carlson is generalized by proving the following inequalities:

$$\left|\sum_{n=1}^{\infty} a_n\right|^q+\left|\sum_{n=1}^{\infty}(-1)^n a_n\right|^q < \tfrac{1}{2}\pi q\left\{\sum_{n=1}^{\infty} n^{\sigma_1}|a_n|^{\sigma_1}\right\}^{1/\sigma_1}\left\{\sum_{n=1}^{\infty}|a_n|^{\sigma_2}\right\}^{1/(\sigma_2-1)\sigma_1},$$

$$\left|\sum_{n=1}^{\infty} a_n\right|^q < \tfrac{1}{2}\pi q\left\{\sum_{n=1}^{\infty}(n-\tfrac{1}{2})^{\sigma_1}|a_n|^{\sigma_1}\right\}^{1/\sigma_1}\left\{\sum_{n=1}^{\infty}|a_n|^{\sigma_2}\right\}^{1/(\sigma_2-1)\sigma_1},$$

where $1\leqq\sigma_1\leqq 2$, $1<\sigma_2\leqq 2$, $q=1+\sigma_2/\sigma_1(\sigma_2-1)$. The following inequalities of opposite character are obtained:

$$\left\{\sum_{n=1}^{\infty}(n-\tfrac{1}{2})^{-1}a_n\right\}\left\{\sum_{n=1}^{\infty}(n-\tfrac{1}{2})^k a_n{}^k\right\}^{(q-1)/(q\nu-1)} > C\left\{\sum_{n=1}^{\infty} a_n{}^\nu\right\}^{q/(q\nu-1)},$$

$$\left\{\sum_{n=1}^{\infty}(2n-1)^{-1}a_{2n-1}\right\}\left\{\sum_{n=1}^{\infty} n^k a_n{}^k\right\}^{(q-1)/(q\nu-1)} > C'\left\{\sum_{n=1}^{\infty} a_n{}^\nu\right\}^{q/(q\nu-1)},$$

where $1<k\leqq 2$, $\nu\geqq 2$, $q=1+k^{-1}$. In the first inequality of the latter group $\{a_n\}$ is simply monotonic, in the second doubly monotonic and $a_n>0$. The constants C and C' depend only on k and ν and can be expressed in terms of gamma functions. *G. Szegö* (Stanford University, Calif.).

Zygmund, A. Proof of a theorem of Littlewood and Paley. Bull. Amer. Math. Soc. 51, 439–446 (1945). [MF 12526]

If we define

$$g(\theta)=\left\{\int_0^1(1-\rho)|\phi'(\rho e^{i\theta})|^2 d\rho\right\}^{\frac{1}{2}}, \qquad 0\leqq\rho<1$$

[Marcinkiewicz and Zygmund, Duke Math. J. 4, 473–485 (1938); Zygmund, Trans. Amer. Math. Soc. 55, 170–204 (1944); these Rev. 5, 230], then

$$\left\{\int_0^{2\pi} g^\lambda(\theta)d\theta\right\}^{1/\lambda}\leqq C_\lambda\left\{\int_0^{2\pi}|\phi(e^{i\theta})|^\lambda d\theta\right\}^{1/\lambda}$$

for all positive λ, with C_λ depending only on λ. This result has been proved by Littlewood and Paley [Proc. London Math. Soc. (2) 42, 52–89 (1936)] for $\lambda>1$. The author gives a simpler proof in three steps. He establishes the relation for $\lambda=2$; then shows that, if it is true for λ_0, it is true for $0<\lambda\leqq\lambda_0$; and, finally, reduces the case $\lambda>2$ to $\lambda\leqq 2$. The third step is the main contribution of the present paper.
František Wolf (Berkeley, Calif.).

Hadwiger, H. Über eine Mittelwertformel für Richtungsfunktionale im Vektorraum und einige Anwendungen. J. Reine Angew. Math. 185, 241–252 (1943). [MF 12105]

Consider a "star" $\mathbf{S}_n$ of vectors $\mathbf{a}_\nu$ $(\nu=1,\cdots,n)$ in s-dimensional space, and a scalar function $\Phi(\mathbf{S}_n,\mathbf{u})$ of $\mathbf{S}_n$ and a unit vector $\mathbf{u}$. To secure additivity, homogeneity, etc., the author confines his discussion to functions of the form

$$\Phi(\mathbf{S}_n,\mathbf{u})=\sum_{\nu=1}^{n}|\mathbf{a}_\nu|^\alpha\chi(\theta_\nu),$$

where $\alpha\geqq 0$, θ_ν is the angle between $\mathbf{a}_\nu$ and $\mathbf{u}$, and $\chi(\theta)$ is bounded and Riemann integrable over $[0,\pi]$. The mean value $M(\mathbf{S}_n)$ of Φ as $\mathbf{u}$ varies is given by

$$\frac{M(\mathbf{S}_n)}{\sum_1^n|\mathbf{a}_\nu|^\alpha}=\frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}s-\frac{1}{2})}\int_0^\pi\chi(\theta)\sin^{s-2}\theta d\theta=\mu(\Phi).$$

It is shown that vector-stars $\mathbf{S}_n$ can be constructed for which Φ is approximately constant. Hence $\mu(\Phi)$ is the greatest lower bound, for varying $\mathbf{S}_n$, of the maximum of

$$\Phi(\mathbf{S}_n,\mathbf{u})\Big/\sum_1^n|\mathbf{a}_\nu|^\alpha.$$

In special cases this proposition yields various previously known results on vector polygons, vector series, etc. A further deduction is the following: if $\sum_1^s a_{\nu\mu}^2=1$ $(\nu=1,\cdots,n)$ the maximum of the geometric mean of the n linear forms

$$|L_\nu(x)|=\left|\sum_1^s a_{\nu\mu}x_\mu\right|$$

on the unit sphere $\sum_1^s x_\mu^2=1$ is not less than

$$\tfrac{1}{2}\exp\left\{s^{-1}-\sum_1^\infty s[2m(2m+s)]^{-1}\right\}.$$

H. P. Mulholland (Beirut).

San Juan, R. Sur le problème de Watson dans la théorie des séries asymptotiques et solution d'un problème de Carleman de la théorie des fonctions quasi-analytiques. Acta Math. 75, 247–254 (1943). [MF 13213]

Carleman raised the question of whether two quasianalytic functions of different classes are identical if they coincide at a point together with all their derivatives. This paper contains the details of the example answering Carleman's question in the negative, announced previously [C. R. Congrès Intern. Math. Oslo, 1936, vol. 2, p. 94]. The question is reduced to a corresponding one for Watson's problem in the theory of asymptotic series. [Mandelbrojt has shown, more generally, that any function with derivatives of all orders can be written as the sum of two functions belonging to (in general, different) quasianalytic classes [Acta Math. 72, 15–29 (1940); these Rev. 1, 297].] *R. P. Boas, Jr.*

San Juan, R. Methods of decomposition in the theory of quasianalytic functions. Revista Univ. Madrid. Ciencias 2, 4 pp. (1942). (Spanish) [MF 12785]

The paper contains some general remarks on problems of the kind considered in the paper reviewed above.
R. P. Boas, Jr. (Providence, R. I.).

Theory of Sets, Theory of Functions of Real Variables

Monteiro, António. Caractérisation de l'opération de fermeture par un seul axiome. Portugaliae Math. 4, 158–160 (1945). [MF 12444]

In a space in which $\bar{O}=O$, the three conditions $X\subset\bar{X}$, $X\subset Y$ implies $\bar{X}\subset\bar{Y}$, and $\bar{\bar{X}}=\bar{X}$ are equivalent to the single condition $Y+\bar{Y}+\bar{\bar{X}}\subset\overline{X+Y}$. *J. F. Randolph.*

Mendonça de Albuquerque, L. The concept of power of sets. Gaz. Mat., Lisboa 4, no. 15, 1–2 (1943). (Portuguese) [MF 12963]

An expository article.

Otchan, G. Sur la permutabilité des opérations δs. Rec. Math. [Mat. Sbornik] N.S. 10(52), 151–163 (1942). (Russian. French summary) [MF 12830]

Given a family N of subsets of the integers, the δs operation F_N defined by the base N carries each sequence $\{E_n\}$

of subsets of a given set into the set

$$F_N E_n = \sum_{s \in N} \prod_{n \in s} E_n.$$

The writer discusses the relationship between this δs operation and the associated "analytic" operation

$$\Phi_N E_n = \sum_{s \in N} \{\prod_{n \in s} E_n \cdot \prod_{n \in s'} E_{n'}\}$$

(where $'$ means complement). He solves the problem of commutativity of such operations; that is, he gives conditions on the two bases N and K which are necessary and sufficient for the relation $F_N F_K E_{nk} = F_K F_N E_{nk}$ to hold for every double sequence $\{E_{nk}\}$ of subsets of a set.

M. M. Day (Providence, R. I.).

Cuesta Dutari, Norberto. Asymmetric continua. Revista Mat. Hisp.-Amer. (4) **4**, 16–23 (1944). (Spanish) [MF 12159]

An ordered set is said to be symmetric (asymmetric) if it is (is not) similar to the set obtained by reversing the order. The author shows that, for any cardinal number greater than or equal to c, there exists a set which is both continuous and asymmetric. The sets constructed make use of the author's previously defined "generalized real numbers" [same Revista (4) **2**, 5–12, 62–66, 104–109, 218–225 (1942); these Rev. **4**, 212]. *J. V. Wehausen.*

Cuesta, N. Dissimilarity of decimal sets. Revista Mat. Hisp.-Amer. (4) **4**, 45–47 (1944). (Spanish) [MF 12161]

This note carries further the author's study of order types by means of "generalized dyadic decimals" [same Revista (4) **3**, 186–205, 242–268 (1943); these Rev. **5**, 231].

J. V. Wehausen (Washington, D. C.).

Fan, Ky. Sur les ensembles monotones-connexes, les ensembles filiformes et les ensembles possédant la propriété des quatre points. Bull. Soc. Roy. Sci. Liége **10**, 625–642 (1941). [MF 13088]

The spaces with which the author is concerned are those of F. Riesz. A set is said to be monotone connected if it is connected, does not consist of a single point, and contains at least one point a_0 with the following property: if F and G are any two connected sets containing a_0 then one of these sets contains the other. The point a_0 is called an extreme point. In order that a set containing two points a_0 and a_1 be irreducibly connected between them it is necessary and sufficient that it be monotone connected and that a_0 and a_1 be extreme points. A connected set containing more than one point is said to be threadlike (filiforme) if, for any three distinct points in it, one of them separates the other two. A point of a threadlike set which does not separate it is called an end. Every monotone connected set is threadlike, and if a threadlike set possesses an end a_0 the set is monotone connected with a_0 as an extreme point. A set E is said to have the property of four points if it is connected, contains more than one point, and if whenever it contains four distinct points a, b, c, d then one of the pairs a and b, a and c, or b and c separates the corresponding remaining pair. In terms of these properties the author characterizes those sets which are topologically equivalent to arcs, rays, lines, and circumferences. *D. Montgomery.*

Fan, Ky. Nouvelles définitions des ensembles possédant la propriété des quatre points et des ensembles filiformes. Bull. Sci. Math. (2) **67**, 187–202 (1943). [MF 12640]

For definitions of threadlike set and the property of four points, see the preceding review. This paper gives necessary and sufficient conditions for these two properties in terms of conditions imposed on connected subsets. For example, it is shown that in order that a connected set containing more than one point possess the property of four points the following condition is necessary and sufficient: if three connected subsets have a point in common then some one of them contains the union of the other two. Several other conditions for the property of four points and for the property of being threadlike are given. These can be lightened when the set is locally connected. In terms of the present results the author obtains new characterizations of "fundamental figures." A fundamental figure is a set which is homeomorphic to a line, a ray, an interval, or a circumference. *D. Montgomery* (Princeton, N. J.).

Choquet, Gustave. L'isométrie des ensembles dans ses rapports avec la théorie du contact et la théorie de la mesure. Mathematica, Timişoara **20**, 29–64 (1944). [MF 12451]

Let T be a closed bounded set in n-space. To every point M of T assume that there corresponds in the plane P (Π) a ray d (δ). A definition of motion of Π on P is then given as follows. To every point M of T the position of Π is determined by bringing the ray δ into coincidence with the ray d. If to every point of M there correspond points a of P and α of Π the image sets of T are called isometric provided they satisfy three local conditions. Besides these definitions there are others, including a generalized definition of rotation without slipping. The instantaneous center of rotation is introduced and all these concepts are studied and amplified in various directions. There is also an investigation of families of lines and their relations to the other topics discussed.

If a numerical function $f(M)$ is defined on a Cartesian set E, this function is said to have derivative zero at M if $\{f(M)-f(M')\}/MM'$ approaches zero when M' approaches M. This concept has applications and there are a number of examples and theorems concerning it. Isometries of varying degrees of strength are introduced and connected with the theory of measure. *D. Montgomery.*

San Juan, Ricardo. Concepts of mathematical analysis. I. Las Ciencias. Madrid **8**, no. 3, 37 pp. (1943). (Spanish) [MF 12783]

An expository account, with extensive bibliographical notes. The topics presented range from the real number system to the theory of integration and trigonometrical series.

San Juan, R. A proof of the theorem of Bolzano-Weierstrass. Revista Univ. Madrid. Ciencias **2**, 4 pp. (1942). (Spanish) [MF 12802]

The theorem in question is not that usually called the Bolzano-Weierstrass theorem in English texts. It states that a real function, upper semicontinuous on a compact set, has a maximum on that set.

Shukla, P. D. On the differentiability of monotone functions. Bull. Calcutta Math. Soc. **37**, 9–14 (1945). [MF 13302]

Conditions for the existence of the derivative of a continuous monotonic function are given in terms of a sequential variable. A typical example is the following theorem. If $P(x)$ is a monotonic nondecreasing continuous function with $P(0)=0$, then $P'_+(0)$ exists provided

$$\lim_{r\to\infty} \{P(x_r)-P(x_{r+1})\}/\{x_r-x_{r+1}\}$$

exists, where $\{x_\nu\}$ is a sequence of positive numbers having the limit zero and $\lim_{\nu\to\infty} x_{\nu+1}/x_\nu = 1$. *P. Civin.*

Brudno, A. Continuity and differentiability. Rec. Math. [Mat. Sbornik] N.S. 13(55), 119–134 (1943). (Russian. English summary) [MF 11649]

This paper is devoted chiefly to proving the following results concerning derivatives of a real function of a real variable, finite at every point. (A) In order that Q be the set of points at which some $f(x)$ does not have a finite (also, finite or infinite) derivative, it is necessary and sufficient that $Q=G_\delta+G_{\delta\sigma}$, where meas $G_{\delta\sigma}=0$. (B) In order that P be the set of points of discontinuity of some function $f(x)$ and Q the set of points where no derivative, finite or infinite, exists, it is necessary and sufficient that (1) $P=F_\sigma$; (2) $Q=G_\delta+G_{\delta\sigma}$, where meas $G_{\delta\sigma}=0$; (3) there exists a set G_δ^* such that $Q\supset G_\delta^*\supset P$. The result (A) had previously been proved by Z. Zahorski with the additional restriction that the points of discontinuity are denumerable [same Rec. N.S. 9(51), 487–510 (1941); these Rev. 3, 73].

J. V. Wehausen (Washington, D. C.).

Tambs Lyche, R. A continuous function without a derivative. Gaz. Mat., Lisboa 4, no. 13, 6–7 (1943). (Portuguese) [MF 12962]

Lebesgue, Henri. A continuous function without a derivative. Gaz. Mat., Lisboa 4, no. 14, 9–10 (1943). (Portuguese) [MF 12965]

Translations of papers in Enseignement Math. 38, 208–211, 212–213 (1942); these Rev. 4, 74.

Popoff, Kyrille. Sur une extension de la notion de dérivée. II. Monatsh. Math. Phys. 51, 115–152 (1944). [MF 12480]

[Part I appeared in the same Monatsh. 48, 103–120 (1939); these Rev. 1, 109.] Let D be the distance of the point $(x+h, f(x+h))$ from the straight line with slope m through the point $(x, f(x))$. For fixed u, denote by $m_1(u)$, $m_2(u)$ the values of m for which $\int_0^u D^2dh$ obtains its minimum or maximum, respectively. As $u\to 0$, $m_i(u)$ tends to a limit if and only if

$$R=\int_0^\infty \{h^2-(\Delta f)^2\}dh \Big/ \int_0^\infty h\Delta f dh$$

has a limit, where $\Delta f = f(x+h)-f(x)$. This limit exists if $f'(x)$ exists and then $m_1(u)$ and $m_2(u)$ tend to the slopes of the tangent and normal. But R may have a limit for more general $f(x)$; for instance, because changing $f(x)$ in a set of measure 0 does not change R. The same idea applied to a variable plane which passes through a point of a space curve yields three stationary planes, which for curves of class 2 coincide with the tangent, normal and rectifying planes of the curve. It can also be applied to surfaces $z=f(x, y)$ and yields generalizations of the tangent plane and the theorems of Meusnier and Euler.

A second part discusses generalizations of derivatives by means of moments and iterations. Let $\varphi(h)$ be continuous and positive. Put $D_0(h)=\Delta f/h$ and

$$D_n(x, u)=\int_0^\infty D_{n-1}(x, h)h\varphi(h)dh \Big/ \int_0^\infty h\varphi(h)dh$$

for $n>0$. If $D_n(f, x)=\lim_{u\to 0} D_n(x, u)$ exists, then $D_{n+i}(f, x)$ exists for $i>0$ and equals $D_n(f, x)$. Thus successive generalizations of derivatives are obtained; D_n satisfies all the fundamental rules of differentiation, including Taylor's theorem. The chain rule holds if $f(\xi)$ has an ordinary derivative and $D_n(g, x)$ exists, where $\xi=g(x)$. The method may also be applied to complex functions and leads to analogues of the Cauchy-Riemann equations. *H. Busemann.*

Besicovitch, A. S. A general form of the covering principle and relative differentiation of additive functions. Proc. Cambridge Philos. Soc. 41, 103–110 (1945). [MF 12841]

The author has previously [Math. Ann. 115, 296–329 (1938)] extended Vitali's covering principle to m-dimensional Hausdorff measure in n-dimensional space; here he establishes a covering theorem for plane sets G which is more general than Vitali's in that plane measure is replaced by any nonnegative additive function of sets, and more special in that the sets covering G are circles $c(x, r)$ such that each point x of G is the center of circles of arbitrarily small radius r. These circles are said to cover G in the Vitali narrow sense. Theorem: if $F(X)$ is a nonnegative additive function of a set X of an additive class Ξ, equal to 0 for any X outside a set G of Ξ, and if a set Γ of circles covers G in the Vitali narrow sense, then Γ contains a subset $\bar{\Gamma}$ of nonoverlapping circles such that $F(\bar{\Gamma})=F(G)$. The proof depends on elementary lemmas concerning the number of circles, such that none contains the center of another, which can intersect a smaller circle.

This covering theorem is applied to the relative derivates of two nonnegative additive functions of sets, F and Φ. Defining $\bar{D}\{x, F/\Phi\}$, $\underline{D}\{\cdots\}$ and $D\{\cdots\}$ as the upper limit, lower limit and limit (if it exists) as $r\to 0$ of $F\{c(x, r)\}/\Phi\{c(x, r)\}$, the author proves that, if U is the set of points for which $D\{x, F/\Phi\}$ does not exist, then $\Phi(U)=0$. As special cases we may take $F(x)$ and $\Phi(x)$ to be the α-dimensional Hausdorff measure of the set X. The paper concludes with examples showing some essential limitations on the generalization of Vitali's principle.

U. S. Haslam-Jones (Oxford).

O'Neill, Anne F. Contributions to the theory of derivates. Duke Math. J. 12, 89–99 (1945). [MF 12072]

The author considers functions of a real variable, taking values in a partially ordered linear space. Using a limiting process defined in terms of the partial ordering, she introduces definitions for continuity, semi-continuity, right- and left-hand upper and lower derivates, etc. She takes first as value space the space (S) of measurable functions of a real variable and shows that, if $F(x)$ is a finite function taking values in (S), there exists for each x_0 a sequence $h_n\downarrow 0$ such that, for example,

$$d^+F(x_0)=\limsup_{n\to\infty}\{F(x_0+h_n)-F(x_0)\}/h_n.$$

If in addition $F(x)$ is continuous, $d^+F(x_0)$ can be expressed in terms of $\{F(x_0+h_n)-F(x_0)\}/h_n$, where $\{h_n\}$ is a fixed enumerable set; for example, the rationals. The measurability of derivates of continuous functions and the relations between them and the partial derivates of the associated functions of two real variables are briefly considered.

The author proves similar results for functions taking values in a space "regular" in the sense of Kantorovitch, remarking that (S) does not completely satisfy this regularity condition. She also considers generalizations of the Baire classification of functions and of results on derivates of convex functions, concluding with some remarks on the relevance of lattice concepts to the theory of derivates.

F. Smithies (Cambridge, England).

Goldenberg, Tudor. On a system of axioms used for the definition of measure of abstract sets. Bull. Math. Soc. Roumaine Sci. **45**, 97–102 (1943). [MF 12760]

The author examines a system of four axioms of Tornier for abstract measure theory. The axioms are shown to be dependent, and an equivalent system of three independent axioms is given. *D. Blackwell* (Washington, D. C.).

Neves Real, Luís. On the algebraic construction of the general theory of measure. Borel measure. Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. 13, 22 pp. (1945) = Anais Fac. Ci. Pôrto **29**, no. 4. (Portuguese. French summary) [MF 13296]

The author shows that, if a Borel measure m is given in a field V_0 of sets, then it can be extended, by a process not involving outer measure, to a Borel measure μ defined over the smallest Borel field V containing V_0. Defining μ by transfinite induction in the natural way, the proof depends on the following lemma. If B in V and $\epsilon>0$ are given, then there exist sets C, H_i and K_i in V_0 such that $C-\sum_i H_i\subset B\subset C+\sum_i K_i$, while $\sum_i m(H_i)<\epsilon$, $\sum_i m(K_i)<\epsilon$ and $m(C)-\epsilon\leqq\mu(B)\leqq m(C)+\epsilon$. *M. M. Day* (Providence, R. I.).

Gomes, Ruy Luís. On an algebraic construction of the notion of integral. Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. 12, 28 pp. (1945) = Anais Fac. Ci. Pôrto **29**, no. 4. (Portuguese. French summary) [MF 13297]

The author studies integration theory in a distributive lattice E with a minimal element and relative complements. Inner and outer measures, m_0 and m^0, are restricted by the following axioms which hold for every x and y of E.

(1) $$m_0(x)\leqq m^0(x),$$

(2^0) $$m^0(x+y)+m^0(xy)\leqq m^0(x)+m^0(y),$$

(2_0) $$m_0(x+y)+m_0(xy)\geqq m_0(x)+m_0(y).$$

Define a in E to be measurable if $m_0(a)=m^0(a)$ and call this common value $m(a)$. (3) If a is measurable and is decomposed into disjoint elements x and y, then

$$m(a)=m_0(x)+m^0(y)=m^0(x)+m_0(y).$$

The principal theorem on this measure is that those measurable elements of E which precede a fixed measurable a of E form a field in which measure is additive. Under additional restrictions on E, if m_0 is nonnegative and if $m^0(\sum x_i)\leqq\sum m^0(x_i)$ whenever the x_i are disjoint elements of E preceding a, then the measurable elements preceding a form a Borel field on which the measure is completely additive. For integration, the author adapts the methods of Carathéodory [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1938, 27–69 (1938)] and Ridder [Acta Math. 73, 131–173 (1941); these Rev. 3, 206]. *M. M. Day.*

Schärf, Henryk. Intégrale et mesure dans certains espaces algébriques. Portugaliae Math. **4**, 211–216 (1945). [MF 13331]

Let G be a semigroup of transformations of a space S into itself. Generalizing a method of Banach, the author proves the existence of a linear functional $I(f)$, defined for all real-valued bounded functions on S, which is nonnegative for nonnegative f, equal to 1 if f is the constant function 1, and such that (a) if G is commutative then $I(f)$ is invariant under the transformations of G, (b) if G is a group and if a function f has in a certain sense a mean value, then $I(f)$ is the mean value, and for such functions is invariant under the transformations of G. As a corollary, there exists in Euclidean n-dimensional space a finitely additive measure which is defined for all subsets, has the value 1 for the whole space, and is invariant under the group of all translations. *L. H. Loomis* (Cambridge, Mass.).

Ríos, Sixto. Lectures on the theory of the integral. Revista Acad. Ci. Madrid **36**, 154 pp. (1942). (Spanish) [MF 12808]

This monograph gives a clear and concise exposition of the theory of the Lebesgue integral, together with a brief introduction to the Denjoy, Perron and Stieltjes integrals. For difficult proofs, references are given to the original articles or other treatments of integration. The section headings are as follows. 1. General properties of sets. 2. Measure of sets. 3. Concept of Lebesgue integral. 4. Integrals before Lebesgue. 5. Measurable functions. 6. Lebesgue integrals of bounded functions. 7. Passage to limit under integral sign. 8. Integrals of unbounded functions. 9. Sets and functions measurable (B). 10. Functions representable analytically. Classes of Baire. 11. Reduction of multiple integrals to iterated integrals. 12. Geometric interpretation of Lebesgue integral. 13. Integrals analogous to Lebesgue integrals. 14. Functions of bounded variation. 15. Indefinite L-integral. Derived numbers. Differentiation of functions of bounded variation. 16. Differentiation of indefinite integrals. 17. Characterization of indefinite integrals. 18. Integration of a derivative. Rule of Barlow. 19. Some properties of the L-integral. 20. General integral of Denjoy, or totalization. 21. Perron integral. 22. Stieltjes integral. 23. Linear functionals. 24. Stieltjes-Lebesgue integral, and integration in abstract spaces.

T. H. Hildebrandt (Ann Arbor, Mich.).

Martinez Salas, J. Note on the Lebesgue set. Revista Mat. Hisp.-Amer. (4) **4**, 234–237 (1944). (Spanish) [MF 12990]

The author shows that a point x at which (*) the set $E_t[f(t)=f(x)]$ has density 1 is a point of the Lebesgue set for $f(x)$, that is, a point at which $\int_x^y|f(t)-f(x)|dt$ has the derivative zero. He attempts to prove, conversely, that the only points of discontinuity (of the second kind) in the Lebesgue set are those satisfying (*). [The proposed theorem is negated by the example $f(t)=t$, $t\neq 1/n$; $f(1/n)=1$ $(n=1, 2, \cdots)$; $x=0$.] *R. P. Boas, Jr.*

Frenkel, Yanny. Properties of nonadditive functions of intervals and their application to the theory of the integral. Revista Union Mat. Argentina **10**, 128–130 (1945). (Spanish) [MF 12506]

Introductory remarks concerning a forthcoming memoir. No results are stated. *R. P. Boas, Jr.*

Massignon, Daniel. Du plan tangent à une surface engendrée par une courbe. Revue Sci. (Rev. Rose Illus.) **79**, 201–208 (1941). [MF 12861]

An attempt to formulate a geometric definition of the tangent plane to a surface, defined as the locus generated by a moving variable curve. *P. Franklin.*

Theory of Series

Erdös, Paul, and Niven, Ivan. On certain variations of the harmonic series. Bull. Amer. Math. Soc. **51**, 433–436 (1945). [MF 12524]

Conditions are given for convergence of special series obtained by changing the signs of blocks of terms of the

harmonic series $\sum n^{-1}$. Let $u_p = p^{-1}$ and let $S(p, q) = u_p + u_{p+1} + \cdots + u_{p+q-1}$. With n and k fixed positive integers, let k_2 be the greatest integer for which $S(n+k, k_2) < S(n, k)$; let k_3 be the greatest integer for which $S(n+k+k_2, k_3) < S(n+k, k_2)$, and so on. The series

$$(1) \qquad S(n, k) - S(n+k, k_2) + S(n+k+k_2, k_3) - \cdots$$

is then an alternating series, say $T_1 - T_2 + T_3 - \cdots$, where $T_p > 0$ and T_p is decreasing. If n and k are such that $k_2 = k$, then $k_j = k$ when $j = 3, 4, 5, \cdots$; in this case, each T_p contains exactly k terms of the harmonic series, so that $T_p \to 0$ and (1) converges. If n and k are such that $k_2 \neq k$ (and hence $k_2 > k$), then the decreasing sequence T_p has a positive limit and (1) diverges. The proofs depend on elementary integral inequalities. *R. P. Agnew* (Ithaca, N. Y.).

Kaplansky, Irving, and Pollard, Harry. Note on the preceding paper. Bull. Amer. Math. Soc. **51**, 437–438 (1945). [MF 12525]

The result of Erdös and Niven, given in the paper reviewed above, that the equality $k_2 = k$ implies $k_2 = k_3 = \cdots$, is generalized to series other than $\sum n^{-1}$. Let $\sum u_n$ be any series whose terms are positive, decreasing, completely monotonic and such that $u_{n+1}/u_n \to 1$ as $n \to \infty$. If $S(n+k, k+1) > S(n, k)$ for a pair of positive integers n and k, then the same inequality holds when n is replaced by $n+1$. The proof uses Stieltjes integral representations of moment sequences. *R. P. Agnew* (Ithaca, N. Y.).

Pollard, Harry. Sequences with vanishing even differences. Duke Math. J. **12**, 303–304 (1945). [MF 12601]

Let $x_0, x_1, \cdots$ be a complex sequence with differences $d_0, d_1, \cdots$ defined by

$$d_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x_k.$$

If $x_k = O(k)$ and $d_n = 0$ when n is even, then $x_k = kx_1$ for all k. The proof is simpler than proofs of related theorems given independently by W. H. J. Fuchs [Proc. Cambridge Philos. Soc. **40**, 189–197 (1944); these Rev. **6**, 46] and R. P. Agnew [Amer. J. Math. **66**, 339–340 (1944); these Rev, **6**, 46] in treatments of Hurwitz-Silverman-Hausdorff methods of summability. *R. P. Agnew* (Ithaca, N. Y.).

Agnew, Ralph Palmer. Abel transforms of Tauberian series. Duke Math. J. **12**, 27–36 (1945). [MF 12067]

Let L denote the set of limit points of the sequence $s_0, s_1, \cdots$ of partial sums of $\sum u_n$. Let L_A denote the set of limit points of the Abel transformation $\sigma(t) = \sum t^k u_k$; $z'' \epsilon L_A$ if there is a subsequence $t_1, t_2, \cdots$ such that $0 < t_n < 1$, $t_n \to 1$, and $\sigma(t_n) \to z''$ as $n \to \infty$. Hadwiger [Comment. Math. Helv. **16**, 209–214 (1944); these Rev. **5**, 236] proved that each of the following assertions is true when $\rho = 1.0160 \cdots$, and false when $\rho < .4858 \cdots$. (1) If $\sum u_n$ satisfies the Tauberian condition $n|u_n| < K$, then to each $z' \epsilon L$ corresponds a $z'' \epsilon L_A$ such that

$$(*) \qquad |z' - z''| \leqq \rho \limsup_{n\to\infty} n|u_n|.$$

(2) If $\sum u_n$ satisfies $n|u_n| < K$, then to each $z'' \epsilon L_A$ corresponds a $z' \epsilon L$ such that (*) holds.

Thus if $nu_n = o(1)$, L and L_A are identical; but if $\limsup n|u_n| = K > 0$, L and L_A may differ. The author proves that, if

$$\rho_1 = \gamma + \log\log 2 + 2\int_{\log 2}^{\infty} x^{-1} e^{-x} dx = .9680 \cdots,$$

then assertions (1) and (2) are both true for $\rho \geqq \rho_1$, while assertion (1) is false for $\rho < \rho_1$, even if the series are restricted to be real. He proves on the other hand that for real series assertion (2) is true for all $\rho \geqq 0$, but shows, like Hadwiger, that in the general case it is false for $\rho < .4858 \cdots$. *L. S. Bosanquet* (London).

Agnew, Ralph Palmer. On cores of bounded divergent complex sequences and of their transforms by square matrices. Revista Ci., Lima **47**, 87–103 (1945). [MF 12708]

Let $s_1, s_2, \cdots$ be a sequence of complex numbers. For each n, let C_n denote the least convex closed set in the finite complex plane containing $s_n, s_{n+1}, s_{n+2}, \cdots$. The set $C = C_1C_2C_3 \cdots$ is called the core of the sequence $\{s_k\}$. By means of several lemmas concerning cores the author proves the following two theorems; in the first the word "real" is omitted and in the second it is included. In order that a method of summability A, defined by a matrix (a_{nk}) of complex constants, may be such that the transform $\{\sigma_n\}$ of each (real) bounded divergent sequence $\{s_k\}$ exists and has a core Γ which is a nonempty subset of the core C of $\{s_k\}$, it is necessary and sufficient that A be regular and that $\lim_n \sum_k |a_{nk}| = 1$. Analogues of these theorems are shown to hold for the class of sequence-to-function transformations. *J. D. Hill* (East Lansing, Mich.).

Brudno, A. Summation of bounded sequences by matrices. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 191–247 (1945). (Russian. English summary) [MF 13009]

Proofs of results previously announced in C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 183–185 (1944); these Rev. **6**, 150. *R. P. Agnew* (Ithaca, N. Y.).

Szász, Otto. On Lebesgue summability and its generalization to integrals. Amer. J. Math. **67**, 389–396 (1945). [MF 12918]

A series $\sum_1^\infty a_n$ is L-summable to the value S if the series $\sum_1^\infty n^{-1} a_n \sin nt = F(t)$ converges in some interval $-\tau < t < \tau$ and $\lim_{t\to 0} t^{-1}F(t) = S$. The author proves that, if $\sum a_n$ is summable $(C, 1-\alpha)$ for some α, $0 < \alpha < 1$, and if $\sum_1^n |S_\nu^{-\alpha}| = O(n^{1-\alpha})$, $n \to \infty$, then the series is L-summable. Several generalizations of L-summability for integrals are suggested, but the inclusion problem is postponed for later study. *H. Pollard* (New Haven, Conn.).

San Juan, R. Differentiation and integration of asymptotic series. Revista Univ. Madrid. Ciencias **2**, 18 pp. (1942). (Spanish) [MF 12801]

The results are similar to the standard ones [for differentiation, see Ritt, Bull. Amer. Math. Soc. **24**, 225–227 (1918)], except that they apply to asymptotic representations in a general region instead of in an angle. *R. P. Boas, Jr.* (Providence, R. I.).

Šklyarskiĭ, D. O. Conditionally convergent series of vectors. Uspehi Matem. Nauk **10**, 51–59 (1944). (Russian) [MF 12671]

The author extends the theorem of P. Lévy and Steinitz by giving an interesting geometric picture of the set S of all possible sums of rearrangements of a conditionally convergent series of vectors in an n-dimensional space R_n. Let U be a series of points of R_n; let V be the smallest closed convex set containing all finite sums of terms of U; let W be the set of all vectors of the form $2v$ with v in V. The principal result is that, if at least one arrangement of the terms of U is convergent, S is the set of all centers of sym-

metry of the set W, and is therefore a k-dimensional flat subset of R_n for some k, $0 \leqq k \leqq n$. A consequence of this is theorem 1: if W is bounded, then U is unconditionally convergent and S contains only the (unique) center of symmetry of W. [Reviewer's comment. The author asks about extension of these results to more general spaces. Theorem 1 can be shown to hold in each sequentially weakly complete Banach space; in particular, it holds in l_1 and in any reflexive space.] *M. M. Day.*

Nicolesco, Miron. Sur les suites doubles. I. Bull. Math. Soc. Roumaine Sci. **42**, no. 1, 53–56 (1940). [MF 12737]

A double sequence s_{mn} $(m, n=1, 2, \cdots)$ is called globally convergent if to each positive ϵ corresponds an integer N such that

$$|s_{mn}-s_{m+p,m}-s_{m,n+q}+s_{m+p,n+q}|<\epsilon$$

when $m, n>N$ and $p, q \geqq 0$. Each sequence which is convergent (in the Pringsheim sense) is globally convergent, but the converse is not true. Each bounded globally convergent sequence contains a convergent double subsequence. A bounded sequence, of which each row and each column is convergent, is convergent if and only if it is globally convergent. [Reviewer's remark. The particular sequence $s_{mn}=(-1)^m+(-1)^n$, which is bounded and globally convergent, contains convergent double subsequences converging to different values.] *R. P. Agnew* (Ithaca, N. Y.).

Sheffer, I. M. Convergence of multiply-infinite series. Amer. Math. Monthly **52**, 365–376 (1945). [MF 13381]

A method for evaluation of double series, different from convergence (Pringsheim), is given. Let $\sum_{i,j=0}^{\infty;\infty} u_{ij}$ be a double series with partial sums $s_{mn}=\sum_{i,j=0}^{m;n} u_{ij}$. The series is summable (or evaluable) S to s, and we write $S(\sum u_n)=s$, if the following is true: to each positive ϵ corresponds a pair of integers p and q such that $|s-\sum'|<\epsilon$ whenever $\sum'$ is a sum of terms u_{ij} such that (1) $\sum'$ contains the term u_{pq} and (2) whenever $\sum'$ contains a term u_{mn}, it contains all terms of the sum s_{mn}. Properties of S summability are obtained by elementary methods.

If $S(\sum u_{ij})=s$ then $\sum u_{ij}=s$, the latter equation meaning as usual that $\sum u_{ij}$ converges to s; but the convergent series whose first two rows are

$$+1-1+1-1+1+\cdots$$
$$-1+1-1+1-1+\cdots$$

and whose lower rows are series of zeros is nonsummable S. Hence S is a nonregular method of summability, being weaker than convergence and included by convergence. However, S is stronger than absolute convergence. Linearity and comparison tests hold for S summability as for convergence. If $S(\sum u_{ij})=s$, then $\sum u_{ij}$ converges by rows to s and converges by columns to s. *R. P. Agnew.*

Herzog, F., and Bissinger, B. H. A generalization of Borel's and F. Bernstein's theorems on continued fractions. Duke Math. J. **12**, 325–334 (1945). [MF 12604]

The classical results of Borel [Rend. Circ. Mat. Palermo **27**, 247–271 (1909)] and F. Bernstein [Math. Ann. **71**, 417–439 (1911)] are generalized to the case of the generalized continued fractions introduced by Bissinger [Bull. Amer. Math. Soc. **50**, 868–876 (1944); these Rev. **6**, 150]. The proofs are elementary but rather intricate.

M. Kac (Ithaca, N. Y.).

Differential Equations

Barroso, Vergílio S. Brief remarks concerning a proof. Gaz. Mat., Lisboa **6**, no. 23, 3–6 (1945). (Portuguese) [MF 12532]

The author discusses the treatment by G. Sansone [Equazioni Differenziali nel Campo Reale, Parte prima, Zanichelli, Bologna, 1941] of an elementary theorem on the existence and uniqueness of a set of solutions of a system of ordinary first order differential equations.

E. F. Beckenbach (Los Angeles, Calif.).

Titchmarsh, E. C. On the eigenvalues of differential equations. J. London Math. Soc. **19**, 66–68 (1944). [MF 12550]

The paper deals with the nth characteristic value λ_n of the differential equation

$$y''+\{\lambda-g(x)\}y=0,$$

for the interval $0 \leqq x<\infty$, with a boundary condition at $x=0$ and $g(0)=0$. It is assumed (i) that $g(x)$ is twice differentiable and $g(x)\to\infty$ as $x\to\infty$; (ii) that, for $x>0$, $g'(x)>0$, $g''(x)\geqq 0$ and $g''(x)=O\{(g'(x))^{4/3}\}$. On this basis the (known) relation

$$\pi^{-1}\int_0^{h_n}\{\lambda_n-g(x)\}^{\frac{1}{2}}dx\sim n,$$

where $g(h_n)=\lambda_n$, is derived by the method of Sturm. It is also shown that, under the condition (i), $\lambda_n=O(n^2)$.

R. E. Langer (Madison, Wis.).

Auluck, F. C., and Kothari, D. S. The quantum mechanics of a bounded linear harmonic oscillator. Proc. Cambridge Philos. Soc. **41**, 175–179 (1945). [MF 12848]

The authors consider the eigenvalues n of the differential equation

$$(1) \qquad \psi''+(n+\tfrac{1}{2}-\xi^2)\psi=0,$$

under the boundary conditions $\psi=0$ at $\pm\xi_0$. The eigenvalues are well known for $\xi_0=\infty$, being the integers. The effect of making ξ_0 finite is to displace the eigenvalues toward higher values. The authors find the three lowest eigenvalues as numerical functions of ξ_0 by finding the zeros of the two independent solutions of (1), which can easily be expressed in terms of confluent hypergeometric functions. Approximate formulas are derived for small values of ξ_0 by using the asymptotic expressions for these functions. Higher approximations to this result and for the case of large ξ_0 are obtained by using perturbation theory.

H. Feshbach (Cambridge, Mass.).

Dieulefait, C. E. On ordinary differential equations with constant coefficients and the operational calculus. An. Soc. Ci. Argentina **139**, 147–151 (1945). (Spanish) [MF 12711]

The solution of a general initial value problem for the ordinary homogeneous differential equations with constant coefficients is expressed as a contour integral. The relation is derived from basic theorems on complex variables.

I. Opatowski (Chicago, Ill.).

Carrier, G. F. On the non-linear vibration problem of the elastic string. Quart. Appl. Math. **3**, 157–165 (1945). [MF 12652]

The motion of an elastic string is considered for the case where changes in tension during the motion cannot be

neglected. This nonlinear problem is handled by the perturbation method using as the parameter a quantity closely associated with the "amplitude" of the motion. An approximation in explicit form is given for motions arising from initial sinusoidal deformations. Motions not confined to a plane as well as the transmission of a localized deformation are considered. *N. Levinson* (Cambridge, Mass.).

Rosenblatt, Alfred. On autoexcited oscillations. I. The galloping of electrical transmission lines. Revista Ci., Lima 47, 33–61 (4 plates) (1945). (Spanish) [MF 12707]

The galloping of an electrical transmission cable under the influence of a wind during conditions favorable to ice formation has been discussed qualitatively as due to aerodynamic lift forces. These forces come into play when the cable cross section is distorted from that of a circle by ice formation. Here the author, using the aerodynamic approach of Joukowski, makes a quantitative investigation of the galloping line. Assuming a cross sectional shape, the lift forces are computed and used in the partial differential equation for the oscillating cable. Conditions for stability are given.

N. Levinson (Cambridge, Mass.).

Minorsky, N. On parametric excitation. J. Franklin Inst. 240, 25–46 (1945). [MF 12691]

The phenomenon of excitation produced by the periodic variation of one or several parameters of a system is referred to as parametric excitation. The author observes that, as has been demonstrated experimentally by Mandelstam and Papalexi, the study of nonlinear differential equations involving parametric excitation is of great practical relevance. However, results to date have been restricted to linear equations of the Mathieu-Hill type. Considering the Hill-Meissner equation $\ddot{q}+(1\pm\gamma)q=0$, $\gamma\ll 0$, the author derives a criterion for the phase and period of the change of sign in the $\pm$ term which indicates whether $q(t)$ will tend to zero, oscillate periodically, or oscillate with divergent amplitude. This criterion is developed by studying solutions in the $(q, \dot{q})$ plane. The criterion is then extended to the case of the Mathieu equation. *N. Levinson.*

Rocard, Yves. Les oscillations de relaxation. Revue Sci. (Rev. Rose Illus.) 79, 31–50 (1941). [MF 12855]

An expository account is given of results of van der Pol and Lienard. Phenomena such as synchronization and frequency demultiplication are discussed. The presentation is oriented to the engineer and physicist. *N. Levinson.*

Shtokalo, I. Méthode asymptotique pour la solution de certaines classes d'équations différentielles linéaires à coefficients variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 51–52 (1945). [MF 12547]

This note refers to a differential system

$$(1)\qquad dx/dt=\{A+\epsilon f(t)\}x,$$

in which A is a constant square matrix of order n, $f(t)$ is a matrix of exponential sums, x is a vector and ϵ is a positive parameter. A certain formal system

$$(2)\qquad \frac{d\xi}{dt}=\left(\sum_{n=0}^{\infty}\epsilon^n A_n\right)\xi,$$

in which the A_n are constants, is related to (1) and a theorem to the following effect is announced. If in the formal development of each Hurwitz determinant for the system (2) the first nonzero coefficient is positive, then for sufficiently small ϵ the solution of the system (1) tends to zero exponentially as $t\to\infty$; but if at least one of those coefficients is negative, the system possesses a solution that is unbounded as $t\to\infty$. *R. E. Langer* (Madison, Wis.).

Bouligand, Georges. Sur les multiplicités caractéristiques de certaines équations aux dérivées partielles. Revue Sci. (Rev. Rose Illus.) 79, 110–112 (1941). [MF 12860]

To generate an integral surface of the first order partial differential equation $f(x, y, z, p, q)=0$ by means of the characteristics of Cauchy, the classical theory requires the existence of the second partial derivatives r, s, t. Starting with a complete integral $z=V(x, y, a, b)$ of $f=0$, one can find the characteristics in the sense of Lagrange without using r, s, t. The author terms the latter generalized characteristics and indicates that they may be looked on as the limiting value of a set of Cauchy characteristics. A method of obtaining an integral surface of $f=0$ without assuming the existence of r, s, t is suggested. Similarly, for second order equations of the Monge-Ampère type, the author points out that the characteristics may be generalized and more integral surfaces obtained. *F. G. Dressel.*

Shapiro, Z. On elliptical systems of partial differential equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 133–135 (1945). [MF 12700]

The system

$$(1)\qquad \sum_{j=1}^{2n}\sum_{k=1}^{3}a_{ij}^{k}\partial u^j/\partial x_k=0,\qquad i=1,\cdots,2n,$$

is called elliptic if the form of order $2n$ in the variables s^1, s^2, s^3 defined by the determinant $|a^1_{ij}s^1+a^2_{ij}s^2+a^3_{ij}s^3|$ is positive definite. If the a^k_{ij} are constants and the system (1) is elliptic, the author considers the boundary value problem for which (a) $u^1, \cdots, u^{2n}$ is a solution of (1) in the half-space $x_3>0$, (b) $\lim_{x_3\to 0}u^i=f_i(x_1, x_2)$, $i=1, \cdots, n$, where $f_i(x_1, x_2)$ are n preassigned continuous functions. To solve the problem the author forms the fundamental solution of (1) in terms of Abelian integrals. He also treats a similar boundary value problem with the half-space $x_3>0$ replaced by a convex region D. Proofs are sketched. *F. G. Dressel.*

Vranceanu, G. Sur les invariants des équations aux dérivées partielles du second ordre. Bull. Math. Soc. Roumaine Sci. 42, no. 1, 91–105 (1940). [MF 12741]

The author gives a systematic classification of second order partial differential equations

$$(1)\qquad F(x, y, z, p, q, r, s, t)=0,$$

based on the characteristics of (1). The two cases, distinct and nondistinct characteristics, are characterized by the invariants of the system of Pfaff

$$dz-pdx-qdy=0,\quad dp-rdx-sdy=0,\quad dq-sdx-tdy=0$$

associated with (1). *F. G. Dressel* (Durham, N. C.).

Pleijel, Åke. Sur la distribution des valeurs propres de problèmes régis par l'équation $\Delta u+\lambda k(x, y)u=0$. Ark. Mat. Astr. Fys. 29B, no. 7, 8 pp. (1943). [MF 12017]

If $k(x, y)>0$ in a domain S of the plane and on its boundary B, the asymptotic distribution of the eigenvalues for $u_{xx}+u_{yy}+\lambda ku=0$ with vanishing boundary values either of u or of its normal derivative is given by the formula

$$4\pi N(t)\sim t\iint_S k\,dxdy,$$

where $N(t)$ denotes the number of eigenvalues below t. This result, first proved by H. Weyl and later, with methods of the calculus of variations, by R. Courant, is generalized in the present paper to the case that $k(x, y)$ changes sign. Denoting by S^+, S^- and S^0, respectively, the subdomains of S where $k>0$, $k<0$ and $k=0$, it is shown that, for either of the two boundary conditions,

$$4\pi N^+(t)\sim t\iint_{S+} k\,dxdy, \quad 4\pi N^-(t)\sim t\iint_{S-} k\,dxdy,$$

where N^+ and N^- refer, respectively, to the number of positive or negative eigenvalues. The proof is based on the definition of the eigenvalues by a sequence of variational problems and proceeds by a generalization of Courant's variational method. *R. Courant* (New York, N. Y.).

Pleijel, Åke. Quelques problèmes de vibrations et les méthodes directes du calcul des variations. Ark. Mat. Astr. Fys. **29A**, no. 23, 17 pp. (1943). [MF 12021]

While the paper reviewed above analyzes the distribution of eigenvalues for $\Delta u+\lambda ku=0$, the present paper establishes the existence of eigenfunctions and eigenvalues by extending and supplementing the theory presented in chap. VII, vol. II, of Courant-Hilbert, Methoden der Mathematischen Physik [Springer, Berlin, 1937], where the existence proof is carried out under the assumption that $k>0$ in S and B. Essentially new arguments and constructions are required for extending certain basic integral inequalities and convergence theorems to the case where k changes sign in S.

At the end of the paper the author calls attention to the fact that the method can be applied to prove the existence of eigenfunctions in more complicated problems; specifically, the treatment of the problem of vibrations of isentropic elastic bodies is briefly indicated. However, the paper does not mention a major, though elusive, difficulty arising for free vibrations; in this case the theory depends on an integral inequality which appraises the Dirichlet integral in terms of the energy integral and leads to the conclusion that the first nontrivial eigenvalue of the problem (defined by a variational problem as a greatest lower bound of the energy integral) is positive. *R. Courant.*

Jacob, Caïus. Sur le problème de la dérivée oblique de Poincaré et sa connexion avec le problème de Hilbert. Bull. Math. Soc. Roumaine Sci. **42**, no. 2, 9–47 (1940). [MF 12743]

Let C be a simple closed Jordan curve with continuous curvature, bounding a finite domain Ω, and let $a(s)$, $b(s)$ and $g(s)$ be continuous functions of the arc length s, each satisfying a Hölder condition. The problems referred to in the title are the following. Poincaré's problem: determine a solution $U(x, y)$ of the equation $a(s)dU/dn-b(s)dU/ds=g(s)$, such that $U(x, y)$ is continuous with its first derivatives in $\Omega+C$ and harmonic in Ω. Here n is the inner normal to C. Hilbert's problem: determine a function

$$f(z)=u(x, y)+iv(x, y),$$

continuous in $\Omega+C$, analytic in Ω and satisfying the equation $a(s)u(s)+b(s)v(s)=g(s)$ on C. Poincaré and Hilbert reduced their problems to the solution of singular integral equations. The author noted that there may be associated with each solution a regular integral equation of Fredholm type and that these integral equations in turn are "associated" in the sense used in the theory of integral equations. He then raised the question of the equivalence of the problems of Poincaré and Hilbert and of the regular integral equations he had associated with them.

The author now considers several special cases of this question; these special cases are related in the same sense in which the problems of Dirichlet and Neumann are related. The discussion is a natural extension of Fredholm's work on the Dirichlet problem. The results are too complicated to quote in detail. *M. O. Reade* (Arlington, Va.).

Lammel, Ernst. Reibungslose Strömung im Aussengebiet eines Kreises und zweier Kreise. Z. Angew. Math. Mech. **23**, 289–291 (1943). [MF 11742]

By mapping the exterior of a circle onto a half-plane and using the Schwarz reflection principle, the author solves simply the problem, previously solved by E. Graeser [Deutsche Math. **1**, 825–858 (1936)], of determining the potential flows having the circumference of the circle as a streamline and having an assigned velocity at infinity. The same method is then applied to the first part of the similar problem for two circles treated earlier by Lagally [same Z. **9**, 299–305 (1929)]. *L. H. Loomis* (Cambridge, Mass.).

Brelot, M. Sur la théorie autonome des fonctions sousharmoniques. Bull. Sci. Math. (2) **65**, 72–98 (1941). [MF 12799]

This paper is complementary to another by the author [Acta Litt. Sci. Szeged **9**, 133–153 (1939); these Rev. **1**, 121]. Extensive use is made of the hypofunctions and hyperfunctions discussed in the earlier paper. A set E in a space of $n\geqq 2$ dimensions is said to be polar if there exists an associated function subharmonic throughout space and negatively infinite on E. Various properties of polar sets are given, including necessary and sufficient conditions for polarity in terms of hyperfunctions. In theorems 3 and 4 the connection between polarity and irregularity with reference to the Dirichlet problem is discussed. [As indicated in a handwritten marginal notation in the reviewer's copy, the second part of theorem 3 is not completely proved; this also affects the demonstration of theorem 4.]

Let E be an arbitrary set and let Ω be a proper bounded open set. If the hypofunction (hyperfunction) for Ω and the characteristic function of E is identically zero then E is said to be interiorly (exteriorly) negligible for Ω. Various properties of such sets are studied. For instance, it is found that the inaccessible frontier points of Ω form a set interiorly negligible for Ω. If a set is interiorly (exteriorly) negligible for every Ω, it is said to be absolutely interiorly (exteriorly) negligible. Any set E of capacity zero is absolutely interiorly negligible and vice versa; if E is closed or the sum of a denumerable number of closed sets then the properties of being polar, absolutely exteriorly negligible and of capacity zero are identical. *F. W. Perkins* (Hanover, N. H.).

Brelot, M. Sur les ensembles effilés. Bull. Sci. Math. (2) **68**, 12–36 (1944). [MF 12800]

This paper is closely related to earlier papers by the author [Acad. Roy. Belgique. Bull. Cl. Sci. **25**, 127–137 (1939); J. Math. Pures Appl. (9) **19**, 319–337 (1940); these Rev. **1**, 238; **3**, 47]. A set E is "thin" (effilé) at a point O, not necessarily in E, if O is isolated from E (from E with O deleted if E contains O) or if there exists a function u subharmonic in the neighborhood of O such that, if M is a point of E distinct from O, then the superior limit of $u(M)$ as $M\rightarrow O$ is less than $u(O)$. If O is not isolated, a necessary and sufficient condition that E be thin at O is that there exist in the neighborhood of O a distribution of positive

masses for which the potential $v(M)$ is finite when $M=O$, but becomes positively infinite when M, distinct from O, approaches O on E. The author also gives other conditions for thinness in terms of the behavior of potential functions.

Using H. Cartan's results on families of potentials [C. R. Acad. Sci. Paris **214**, 994–997 (1942); these Rev. **5**, 146] the author generalizes an earlier definition of extremalization so that, given any function u subharmonic in the open set Ω, and any bounded set E contained, together with its limit points, in Ω, the extremal V of u for E is uniquely determined. Here V is subharmonic in Ω, $V\geqq u$ in Ω, and $V=u$ in $\Omega-E$ except possibly on a polar set including only boundary points of E. For given u and E, V is independent of the choice of a suitable Ω. Extremalization includes the sweeping-out process as a special case. A necessary and sufficient condition that u be thin at O is that the extremal of the function $u(M)=OM$ for E be positive at O. Other theorems are given concerning the relation of thinness to extremalization. The relation of thinness to capacity is also studied. Finally the idea of a "pseudo-limit" is introduced: essentially the usual definition of the limit of a function at O, with the requirements waived on a set thin at O. A number of results are given involving pseudo-limits and the solution of the generalized Dirichlet problem.

F. W. Perkins (Hanover, N. H.).

Petrashen, G. I. Solutions of vector boundary problems of mathematical physics for the sphere. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 266–269 (1945). [MF 12947]

It is well known that from the spherical harmonics $Y_{l,m}(\theta, \varphi)$ it is possible to construct three vectors which satisfy orthogonality relations on the surface of a sphere. An arbitrary vector can be expanded in terms of these vectors. The author makes use of this fact to set up general vectors whose divergence or whose curl is zero. Using the same procedure, he discusses the solution of the time dependent vector wave equation. *H. Feshbach.*

Davids, Norman. Calculation of vertical component (Z) for potential fields from observed values of declination (D) and horizontal intensity (H). Terr. Magnetism **49**, 239–242 (1944). [MF 11548]

When D and H are known over the whole earth it is possible to derive the values of Z. A formula of the type

$$Z(P)=\int_0^\pi G(\theta)\bar{X}d\theta$$

is recommended, where $2G(\theta)=\cos\theta+\csc\frac{1}{2}\theta$ and

$$2\pi\bar{X}=\int_0^{2\pi} H\cos D\,d\phi,$$

θ being the colatitude and ϕ the longitude referred to the given point P as pole. The formula is proved by means of a representation of the scalar potential of the field as a series of the form $\sum(a/r)^{n+1}S_{n,m}(\theta, \phi)$, in which the coefficients are spherical harmonics. A table of $G(\theta)$ with two decimal places is given for $\theta=0°(5°)180°$ and a trial computation is made for the eccentric-dipole field. *H. Bateman.*

Evrard, Pierre. Quelques remarques au sujet de la courbe d'anomalie magnétique verticale due à une couche mince inclinée. Bull. Soc. Roy. Sci. Liége **11**, 105–119 (1942). [MF 13095]

Evrard, Pierre. Les courbes d'anomalies magnétiques horizontale et verticale dues à une couche mince inclinée. Bull. Soc. Roy. Sci. Liége **12**, 103–108 (1943). [MF 13133]

Temliakov, A. Harmonic functions and solutions of the wave equation with three independent variables. Rec. Math. [Mat. Sbornik] N.S. **14(56)**, 133–154 (1944). (Russian. English summary) [MF 12294]

As proved by Bergman [C. R. Acad. Sci. Paris **205**, 1198–1200 (1937); Math. Ann. **99**, 629–659 (1928); **101**, 534–558 (1929)], for every partial differential equation L of elliptic type there exist operators of the form $u=\Re\{\int fdt\}$ which transform the totality A of analytic functions f of a complex variable into solutions of L. Using these operators it is possible to "translate" various relationships in the theory of analytic functions of a complex variable into theorems on (real) solutions of the equation L. This method yields, among others, theorems on the connection between the coefficients of the development of a solution u of L and the location of the singularities of u.

In particular, the use of Hadamard's theorems on the distribution of the ~~poles~~ zeros of an entire function of a complex variable yields corresponding results on solutions of L. In the case where L is a differential equation of elliptic type in two variables results of this kind were obtained by Nielsen [Duke Math. J. **11**, 121–137 (1944); these Rev. **5**, 204] and by Bergman [see the following review]. In continuation of his previous papers [C. R. Acad. Sci. Paris **200**, 799–801 (1935); Rec. Math. [Mat. Sbornik] (42), 707–716 (1935)], and using the same approach, the author gives a procedure for the determination of the location of the poles $\{\sum_{k=1}^3(x_k-a_{k\nu})^2\}^{-\frac{1}{2}}$ of a harmonic function $u(x_1, x_2, x_3)$ in terms of the coefficients of the development of u under assumption that u has these poles as its only singularities and satisfies some additional conditions. His results are consequences of the following theorem. Let

$$\sum_{n=0}^{\infty}\left\{\sum_{m=-n}^{n} n!i^{|m|}((n+|m|)!)^{-1}A_{n,m}e^{m\varphi i}\right\}R^nP_n^m(\cos\theta)$$

be the development of a harmonic function u and let

$$\lim_{n\to\infty} A_{n+1,m+k}/A_{n,m}=\alpha_{m,m+k}, \qquad k=-1, 0, 1.$$

If, for every m,

$$\begin{vmatrix} \alpha_{0,-1} & \alpha_{0,0} & \alpha_{0,1} & 1 \\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & 1 \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & 1 \\ \alpha_{m,m-1} & \alpha_{m,m} & \alpha_{m,m+1} & 1 \end{vmatrix}=0$$

and some other conditions are satisfied, then

$$u=c_0\left\{\sum_{k=1}^{3}(x_k-a_{k,1})^2\right\}^{-1}+u_1,$$

where

$$\sum a_{k,1}^2=R^2, \quad 1/R=\limsup_{n\to\infty}(|A_{n+1,m}|)^{1/n},$$

$m=-(n+1), -n, n, n+1$, and u_1 is regular in $\sum x_k^2\leqq R^2$. The author determines the $a_{k,1}$ from the $\alpha_{0,k}, \alpha_{1,k}, \alpha_{2,k}$, $k=0, -1, 1$. Similar results are obtained for functions satisfying the wave equation. [The reviewer would like to remark that similar theorems can be obtained if u possesses other types of singularities, for example, branch lines along circles.] *S. Bergman* (Providence, R. I.).

Bergman, Stefan. Certain classes of analytic functions of two real variables and their properties. Trans. Amer. Math. Soc. **57**, 299–331 (1945). [MF 12559]

The author continues his investigation of the relation between certain classes of analytic functions of two real variables, defined as solutions of partial differential equa-

tions, and analytic functions of one complex variable. [See, for example, Bull. Amer. Math. Soc. **50**, 535–546 (1944); these Rev. **6**, 2.] Consider a partial differential equation

$$(*)\qquad L(U)\equiv U_{xx}+U_{yy}+AU_x+BU_y+CU=0,$$

where A, B, and C are analytic functions of the real variables x, y in a neighborhood of the origin. Let $E(z, \bar{z}, t)$ be a suitably restricted function and define the integral operator

$$u(z, \bar{z})=P(f)=\int_{-1}^{1}E(z, \bar{z}, t)f(\tfrac{1}{2}z(1-t^2))(1-t^2)^{-\frac{1}{2}}dt.$$

In his earlier papers the author has shown, under suitable restrictions on E, that $U=\Re(u)$ is a solution of (*). Thus P is an operator which transforms an analytic function f into a solution of (*). The author considers the question of determining all operators which transform analytic functions into solutions of (*). By use of infinite matrices he determines a wide class of integral operators having the desired property. He denotes by $C(E)$ the class of complex solutions $u=U+iV$ of (*) whose generating function is of a special type. He then investigates certain similarities of functions of class $C(E)$ to analytic functions of a complex variable; for example, their singularities and value distributions. *W. T. Martin* (Syracuse, N. Y.).

Bergman, Stefan. A class of nonlinear partial differential equations and their properties. Bull. Amer. Math. Soc. **51**, 545–554 (1945). [MF 12816]

In earlier papers the author has considered the relationship which exists between a certain subclass C of complex solutions of a linear differential equation of elliptic type and analytic functions of a complex variable [cf. the preceding review]. The class C is closed under addition but not under multiplication. In the present paper the author defines the class N of functions whose logarithms belong to C. Obviously N is closed under multiplication. If the differential equation giving rise to the class C has the form

$$\Phi_{z\bar{z}}+a\Phi_z+\bar{a}\Phi_{\bar{z}}=0,$$

then the functions of the corresponding class N will satisfy the nonlinear differential equation

$$N(U)\equiv U_{z\bar{z}}+aU_z+\bar{a}U_{\bar{z}}-U_zU_{\bar{z}}/U=0.$$

(Thus functions of class N occur, in particular, in the theory of the two-dimensional heat equations.) The author obtains an analogue of the Poisson-Jensen formula for functions satisfying the equation $N(U)=0$. Other results are obtained for solutions of a related class of differential equations. *W. T. Martin* (Syracuse, N. Y.).

Frankl, F. On Cauchy's problem for partial differential equations of mixed elliptico-hyperbolic type with initial data on the parabolic line. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 195–224 (1944). (Russian. English summary) [MF 12298]

The author considers the problem of determining a function $z(x, y)$, defined for nonpositive values of y of small modulus, which satisfies the differential equation with analytic coefficients

$$(*)\qquad yz_{xx}+z_{yy}+a(x, y)z_x+b(x, y)z_y+c(x, y)z=0$$

and initial conditions of the Cauchy type on the x-axis. Since the prescribed values $\tau(x)=z(x, 0)$, $\nu(x)=z_y(x, 0)$ are not assumed to be analytic the Cauchy-Kowalewski theorem is not applicable. The author first introduces characteristic coordinates $\lambda=x-\frac{2}{3}(-y)^{\frac{3}{2}}$, $\mu=x+\frac{2}{3}(-y)^{\frac{3}{2}}$, so that the singular line goes over into the line $\mu=\lambda$. Then he solves the corresponding problem for the domain $\mu=\lambda+\epsilon$ and finally obtains the solution of the original problem by the limit process $\epsilon\rightarrow 0$. To justify this passage to the limit a careful study of the Riemann function for equation (*) (written in characteristic coordinates) is needed. This function is represented in the form $u=u_0+u_1+\cdots$, where u_0 is the Riemann function of the equation (**) $yz_{xx}+z_{yy}=0$ considered by Darboux, and u_ν ($\nu=1, 2, \cdots$) are defined by recurrence formulas involving indefinite integrals with the kernel u_0. Since u_0 can be represented by a hypergeometric series, the study of u can be based on known properties of hypergeometric functions. The author's final result asserts the existence of a solution z which can be represented in the form

$$z(\lambda_0, \mu_0)=\int_0^1 g(\lambda_0, \mu_0, t)\tau\{\lambda_0+(\mu_0-\lambda_0)t\}dt + \int_0^1 h(\lambda_0, \mu_0, t)\nu\{\lambda_0+(\mu_0-\lambda_0)t\}dt,$$

where

$$g=K_1t^{-5/6}(1-t)^{-5/6},\quad h=K_2t^{-1/6}(1-t)^{1/6}(\mu_0-\lambda_0)^{\frac{1}{3}}$$

and the functions K_1, K_2 are bounded for all values of μ_0, λ_0, t. (In the case of equation (*), K_1 and K_2 are constants.) The nonhomogeneous equation of the form (*) is also discussed. The more general equation

$$f(y)z_{xx}+z_{yy}+az_x+bz_y+cz=0,\quad f(0)=0,\ f'(0)>0,$$

can be reduced to the form (*) by the substitution

$$y'=\int_0^y \tfrac{1}{2}\{f(y)\}^{-\frac{1}{2}}dy.$$

It contains as a special case Chaplygin's hodograph equation for the potential flow of a compressible fluid, which changes its type from elliptic to hyperbolic when the flow becomes supersonic. *L. Bers* (Syracuse, N. Y.).

Karimoff, Dj. Sur les solutions périodiques des équations différentielles non-linéaires de type parabolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 175–178 (1945). [MF 12697]

The author formerly gave the solution of the following periodic boundary value problem:

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial z}{\partial x}\right\}=\Phi(x, t)+\mu f(z),\qquad p(x)>0$$

[same C. R. (N.S.) **28**, 403–406 (1940); these Rev. **2**, 204]. He now treats the same type of problem with the function $f(z)$ replaced by $f(z, \partial z/\partial t)$. The method of successive approximations is used and uniqueness of the solution is proved for μ sufficiently small. *F. G. Dressel* (Durham, N. C.).

Shklover, A. M. Use of complex numbers in solving problems of heat transfer by plane thermal waves. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 106–110 (1944). [MF 12576]

The author considers one-dimensional flow of heat through composite walls with temperatures varying in a simple harmonic manner with respect to time. He presents a short method of determining the damping factor, that is, the ratio of the amplitudes of the temperature variation at the two faces of a layer, and the shift in phase of the temperature variation. The method consists of the direct use of a complex function which satisfies the heat equation. A single

complex number gives the total damping factor and phase-shift for the composite wall. The author considers cases in which some of the layers consist of air. He also considers different types of boundary conditions and a case in which a periodic source of heat is present in air enclosed by a wall.
R. V. Churchill (Ann Arbor, Mich.).

Green, A. E. Double Fourier series and boundary value problems. Proc. Cambridge Philos. Soc. **40**, 222–228 (1944). [MF 11869]

This represents one of the latest attempts to solve the eigenvalue problems for a clamped rectangular plate by expanding the eigenfunctions in series. A typical case in the theory of the buckling of a plate is given by the equation (*) $\Delta\Delta w+\lambda\Delta w=0$, with the boundary conditions (**) $w=dw/dn=0$. In earlier attempts the eigenfunctions w were expanded in series of particular solutions of (*). G. I. Taylor called these series "formal solutions," probably indicating that they were not completely satisfactory [Z. Angew. Math. Mech. **13**, 147–152 (1933)]. The author expands the eigenfunctions w and their derivatives in double Fourier series, substitutes these series in (*) and obtains, by using (**), an infinite set of linear equations for the unknown coefficients. He states that a nonzero solution is obtained if the corresponding infinite determinant vanishes, and that his formulae coincide with those obtained by A. Weinstein [J. London Math. Soc. **10**, 184–192 (1935)], in which no infinite determinants were used. [According to a communication from J. A. Jenkins to the reviewer, the standard test for the convergence of the infinite determinant fails. Moreover, could this convergence be proved, it would remain to prove the convergence of the least zeros of the principal minors. Even if these difficulties were to be overcome, the question arises, in view of the considerable numerical work involved, whether the procedure would be of interest in applications, since it would not provide any information about the error in the approximations.]
A. Weinstein (Toronto, Ont.).

Theory of Probability

Linés Escardó, E. The problem of coincidences. Revista Mat. Hisp.-Amer. (4) **1**, 202–214 (1941). (Spanish) [MF 12989]

Given two decks with p_i cards marked i ($i=1, 2, \cdots$), the problem is to find the probability that in a random permutation of the decks a specified number of matches will occur. The author's method of solution was extended in a later paper to the case where the decks need not have the same structure [same Revista (4) **4**, 188–205 (1945); these Rev. **6**, 232].
I. Kaplansky (Chicago, Ill.).

Onicescu, O. Les structures planes. Bull. Math. Soc. Roumaine Sci. **45**, 63–76 (1943). [MF 12757]

An expository article introducing, for the purposes of the theory of probability, two-dimensional distribution functions and their characteristic functions. The method is similar to that of de la Vallée Poussin using dyadic nets of rectangles.
W. Feller (Ithaca, N. Y.).

Dor, Léopold. Quelques remarques sur les variables aléatoires combinées xy, $\sqrt{\alpha^2x^2+\beta^2y^2}$. Bull. Soc. Roy. Sci. Liége **13**, 203–209 (1944). [MF 13177]

After some formal remarks concerning the distribution function of the product of independent random variables, the author proves that any Gaussian variable z_0 can be represented in the form $c_0^2z_0^2=\sum_1^n c_i^2z_i^2$, where the z_i are again Gaussian and the means m_i and deviations σ_i satisfy $c_0^2m_0^2=\sum_1^n c_i^2m_i^2$, $1-2\sigma_0c_0^2=\prod_1^n\{1-2\sigma_i^2c_i^2\}$, $2c_i^2\sigma_i^2\leqq 1$.
W. Feller (Ithaca, N. Y.).

Haldane, J. B. S. Chance effects and the Gaussian distribution. Philos. Mag. (7) 36, 184–185 (1945). [MF 13341]

The author remarks that Silberstein's paper [Philos. Mag. (7) **35**, 395–404 (1944); these Rev. **6**, 88] "ignores a good deal of earlier work." He then determines the accuracy of the normal approximation to the n-fold convolution of the rectangular distribution with itself.
W. Feller.

Gnedenko, B. V. On the theory of Geiger-Müller counters. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **11**, 101–106 (1941). (Russian) [MF 13371]

Registrations by the counter are treated as random events. After each registration the counter is locked for a fixed time interval during which no registration takes place. The author describes a rather complicated way of computing the expected number and the variance of registrations. If the events are distributed according to a Poisson law, the expressions simplify and at least the expected number $E(t)$ of registrations can be represented explicitly.
W. Feller (Ithaca, N. Y.).

Kurbatov, J. D., and Mann, H. B. Correction of G-M counter data. Phys. Rev. (2) **68**, 40–43 (1945). [MF 12876]

The main result of the paper reviewed above is obtained in a simpler way from an integral equation. The expression for $E(t)$ which the authors obtain is slightly simpler than Gnedenko's [which is copied incorrectly in the paper] but stili complicated. The authors therefore investigate the maximum deviation of $E(t)$ from a linear function.
W. Feller (Ithaca, N. Y.).

Orts, J. M.* On some sequences of random variables. Revista Mat. Hisp.-Amer. (4) **5**, 53–57 (1945). (Spanish) [MF 12993]

Let $p_j>0$, $\sum p_j=1$ ($j=1, \cdots, m$). Define $a_{i,k}$ recursively by $a^i_{i,k+1}=\sum p_ja^i_{j,k}$ ($i=1, \cdots, m$; $k=1, 2, \cdots$). It is shown that $\lim_{k\to\infty} a_{i,k}=a$ exists and is independent of i and the p_j.
W. Feller (Ithaca, N. Y.).

Uspensky, J. V. On the problem of the ruin of gamblers. Publ. Inst. Mat. Univ. Nac. Litoral **7**, 155–186 (1945). (Spanish) [MF 12616]

The author considers the classical problem of ruin. The general solution is based on that for the special case of an infinitely rich adversary, for which four solutions are given.
D. Blackwell (Washington, D. C.).

Hsu, P. L. On the approximate distribution of ratios. Ann. Math. Statistics **16**, 204–210 (1945). [MF 12912]

Continuing his investigations [same Ann. **16**, 1–29 (1945); these Rev. **6**, 233], the author derives asymptotic expansions of the Cramér type for the random variables S^*/S and X^*/S, with

$$S=\{X_1+\cdots+X_m\}/m,\quad S^*=\{X_1^*+\cdots+X_n^*\}/n;$$

here the X_i^* are equidistributed, the X_k are equidistributed and positive, and the variables are mutually independent.
W. Feller (Ithaca, N. Y.).

* The reviewer stated that an error in Cramér's book had not, to the best of his knowledge, been rectified in print. It has subsequently come to his attention that a correction has been published by Cramér [Neuvième congrès des mathématiciens scandinaves, 1938, Helsingfors, 1939, pp. 67-86, in particular pp. 77]

Gnedenko, B. V. Limit theorems for sums of independent random variables. Uspehi Matem. Nauk 10, 115–165 (1944). (Russian) [MF 12673]

A unified treatment of various necessary and sufficient conditions for the weak law of large numbers and the central limit theorem. Most of the results are known but are presented here in a systematic and somewhat simplified manner.

M. Kac (Ithaca, N. Y.).

*Gnedenko, B. V. **Elements of the theory of distribution functions of random vectors.** Uspehi Matem. Nauk 10, 230–244 (1944). (Russian) [MF 12675]

This is an exposition of the salient points of the theory of multidimensional random variables (random vectors), including the treatment of the generalization to vectors of the central limit theorem. [The author rectifies an error in Cramér's book [Random Variables and Probability Distributions, Cambridge University Press, 1937]. The error consists in asserting that the uniform convergence of characteristic functions in a finite interval insures convergence of the corresponding distribution functions. Actually, one must also assume that the characteristic functions converge on the whole real axis. This additional assumption, together with the assumption of uniform convergence in a finite interval, implies uniform convergence in every finite interval. Although this error has been noticed by many readers, it has never, to the best of the reviewer's knowledge, been rectified in print. The author shows the insufficiency of Cramér's condition by constructing a simple counterexample on pp. 237–238.] *M. Kac* (Ithaca, N. Y.).

Smirnov, N. V. Approximate laws of distribution of random variables from empirical data. Uspehi Matem. Nauk 10, 179–206 (1944). (Russian) [MF 12674]

The author gives a thorough discussion of relations between theoretical distributions and "empirical distributions" obtained through consideration of random samples. These relations are obtained in the form of rather interesting limit theorems. The methods and results are closely related to those reviewed previously [Rec. Math. [Mat. Sbornik] N.S. 6(48), 3–26 (1939); Bull. Math. Univ. Moscow 2, no. 2 (1939); these Rev. 1, 246, 345]. *M. Kac* (Ithaca, N. Y.).

Daniels, H. E. The statistical theory of the strength of bundles of threads. I. Proc. Roy. Soc. London. Ser. A. 183, 405–435 (1945). [MF 12692]

Suppose that the strength of a thread is a random variable, so that there exists a probability $b(S)$ that the thread will break if a load S is applied. Consider a bundle consisting of n threads whose strengths are independent random variables $\xi_1, \cdots, \xi_n$ with the same distribution function $b(S)$. If a load S is applied, each thread will have to carry the load S/n. Some threads may break and the load will then be distributed equally among the surviving threads. Thus the bundle will not break under load S if, and only if, there exists an integer $k \leqq n$ such that k among the ξ_i exceed S/k. The probability B_n that the strength of the bundle does not exceed S can therefore be expressed as an n-tuple integral. In studying it, the author arrives at the following new limit theorem which holds for any sequence of nonnegative independent random variables ξ_k with the same distribution function $b(S)$. Suppose that $1-b(S)=o(S^{-1})$ and that $S\{1-b(S)\}$ assumes its maximum for $S=S_0$. Let $\{\sigma_1, \cdots, \sigma_n\}$ be the sequence $\{\xi_k\}$ rearranged in descending order. Let $B_n(S)$ be the probability of the simultaneous realization of the inequalities $k\sigma_k \leqq S$, $k=1, \cdots, n$. Then, as $n\rightarrow\infty$, B_n approaches the normal distribution with mean $nS_0[1-b(S_0)]$ and variance $nS_0b(S_0)[1-b(S_0)]$. The proof depends on a detailed analysis of the asymptotic behavior of

$$T_{n,m,r}=\binom{n-r}{n-m}(\lambda b_m)^{n-m}(1-\lambda b_m)^{m-r},$$

where $n\rightarrow\infty$ and m, r, λ depend on n. The foundations of a more general theory are outlined. *W. Feller.*

Mihoc, G. Sur le problème des itérations dans une suite d'épreuves. Bull. Math. Soc. Roumaine Sci. 45, 81–95 (1943). [MF 12759]

Let λ_n be the number of runs assumed by the variables of a Markov chain whose variables can assume only a finite number of values. The first and second moments of λ_n are evaluated and the asymptotic normality of λ_n is studied, using characteristic functions. *J. L. Doob.*

Beboutoff, M. Markoff chains with a compact state space. Rec. Math. [Mat. Sbornik] N.S. 10(52), 213–238 (1942). (English. Russian summary) [MF 12832]

The author extends the Kryloff-Bogolioubоff analysis of flows in measure spaces [Ann. of Math. (2) 38, 65–113 (1937)] to Markov probability processes. The transition probability function $P(E|x)$, the probability of going from a point x into a set E, where x varies in a compact metric space, is assumed to be such that $\int\varphi(y)P(dy|x)$ is continuous in x for every continuous $\varphi(x)$. It is shown that there is always at least one probability measure on x-space such that the process with these initial probabilities is temporally homogeneous. A complete analysis is made of the possible probability measures with this property, the division of x-space into invariant ergodic sets and related subjects.

J. L. Doob (Glen Echo, Md.).

Fortet, Robert. Sur la notion de fonction aléatoire. Revue Sci. (Rev. Rose Illus.) 79, 135–139 (1941). [MF 12857]

This (partly expository) paper discusses the general notion of stochastic processes, the definition of random variables and the corresponding measure in functional spaces, etc. Applications are given to the results reviewed below.

W. Feller (Ithaca, N. Y.).

Fortet, Robert. Les fonctions aléatoires du type de Markoff associées à certaines équations linéaires aux dérivées partielles du type parabolique. J. Math. Pures Appl. (9) 22, 177–243 (1943). [MF 12334]

[For preliminary reports cf. C. R. Acad. Sci. Paris 212, 325–326, 1118–1120 (1941); 213, 553–556 (1941); these Rev. 3, 4; 5, 125, 123; and the preceding review]. It is well known that a parabolic equation

$$(1) \qquad u_t+u_{xx}+bu_x=0$$

defines a Markov process whose transition probability-densities $U(t, x; \tau, \xi)$ (with $\tau>t$) are given by the elementary solution of (1). Generalizing results of various authors, in particular of Petrovski and P. Lévy, the author investigates this Markov process from two aspects and applies the result of the probabilistic theory to derive a remarkably general existence theorem for (1). It is assumed that $b(t, x)$ and b_x are bounded, and that b_x satisfies a two-dimensional Lipschitz condition. First the author uses known properties of $U(t, x; \tau, \xi)$ to prove that the random variable $X(t)$ representing the Markov process is continuous with probability one. More precisely, to every $c>1$ (but to no $c<1$) there corresponds a $\delta>0$ which is independent of t', t'' and such

that $|t'-t''|<\delta$ implies

(2) $$|X(t')-X(t'')|<2c\{|t'-t''|\log|t'-t''|^{-1}\}^{\frac{1}{2}}.$$

The point of departure of the second part is the measure theory of functional spaces and a transformation which maps $X(t)$ onto a random variable $Y(t)$ associated with the classical N. Wiener case, that is to say (1) with $b=0$. Let the one-valued continuous function $x(t)$ define the curve C, and let D be the domain $x<x(t)$, $t<\tau$. It is shown that in a rigorous sense the following conditional probabilities ("absorption probabilities") exist, all evaluated on the hypothesis that $X(t)=x\leqq x(t)$:

(3) $$\begin{cases} \bar{P}_C(t,x;\tau)=\Pr\{X(t')\leqq x(t') \text{ for all } t\leqq t'\leqq\tau\}, \\ \Phi_C(t,x;\tau)=\Pr\{X(t')=x(t') \text{ for some } t<t'\leqq\tau\}, \\ \bar{\Pi}_C[t,x(t);\tau]=\Pr\{X(t')\geqq x(t') \text{ for all } t\leqq t'\leqq\tau\}. \end{cases}$$

The curve C is called of upper class (supérieure) at $(t,x(t))$ if $\bar{P}_C(t,x(t);t)=1$; of lower class if $\bar{\Pi}_C[t,x(t);t]=1$; of the separating class (séparatrice) if $\bar{P}_C(t,x(t);\tau)=\bar{\Pi}_C[t,x(t);\tau]=0$. The plausible but important result that no other cases are possible is proved, and simple criteria are given. For example, C is not of upper class at $(t,x(t))$ if there exists a sequence $t_n\downarrow t$ such that

(4) $$\limsup|x(t_n)-x(t)|(t_n-t)^{-\frac{1}{2}}<\infty.$$

Among many continuity properties of (3) it is shown that both $\bar{P}_C$ and Φ_C are, as functions of t and x, solutions of (1). An investigation of their boundary values leads to the following existence theorem. Suppose that C is of upper class at no point (for example, suppose that (4) holds). Let $f(t)$ and $g(y)$ be continuous. Then there exists a solution $u(t,x)$ of (1) which is regular in D and such that $u(t,x)\rightarrow f(t)$ as $x\rightarrow x(t)$ with $t<\tau$ fixed and $u(t,x)\rightarrow g(\xi)$ as $t\rightarrow\tau$ with $x\rightarrow\xi<x(\tau)$. Such a solution can be represented by means of Stieltjes integrals involving Φ_C; however, the conditions do not imply uniqueness. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

Fréchet, Maurice. Sur la correspondance entre certaines lois d'erreurs et certaines définitions de la distance. Revue Sci. (Rev. Rose Illus.) 79, 3–14 (1941). [MF 12853]

The paper is in the main a reproduction of a lecture given in 1922. It contains criticisms of attempts to "prove" the universality of the normal distribution such as were usual in the older literature; a discussion of empirical verifications or refutations of the normal distribution; and finally a discussion of error laws to which one is led if the arithmetic mean of observations is replaced by some other mean. *W. Feller* (Ithaca, N. Y.).

Kaplansky, Irving. A common error concerning kurtosis. J. Amer. Statist. Assoc. 40, 259 (1945). [MF 12543]

The following is quoted from the paper. In many texts it is stated that a frequency curve with positive kurtosis is higher in the neighborhood of the mean than the corresponding normal curve, while one with negative kurtosis is lower. In the hope of clearing up this error, we offer in this note four examples showing that any combination of peakedness at the mean and kurtosis may occur. *Z. W. Birnbaum* (Seattle, Wash.).

Delgleize, A. Sur le schéma simple des urnes. Bull. Soc. Roy. Sci. Liége 11, 398–403 (1942). [MF 13111]

Plot the binomial frequency curve as a step-polygon. On the side with $k\leqq x\leqq k+1$ select a point A_k with abscissa $k+f(k)$. Let B_k be the intersection of the line $A_{k-1}A_k$ with $x=k$. The slope of $A_{k-1}A_k$ can be expressed in terms of the coordinates of B_k and $f(k)$. The relation thus obtained can be extended to all x and yields in general a difference-differential equation. For particular choices of $f(x)$ the author finds the Gaussian and the Pearson curves. *W. Feller* (Ithaca, N. Y.).

Delgleize, A. Sur les courbes de fréquence. Bull. Soc. Roy. Sci. Liége 12, 264–276 (1943). [MF 13142]

The formal procedure reviewed above is extended to arbitrary frequency curves. *W. Feller* (Ithaca, N. Y.).

Dehalu, M. Sur la démonstration de la formule de K. Pearson dans le cas du schéma simple des urnes. Bull. Soc. Roy. Sci. Liége 11, 146–151 (1942). [MF 13096]

On sait que Pearson a substitué au calcul approché de la loi binomiale l'équation différentielle de la courbe tangente au milieu des côtés du polygone binomial. Je me propose de démontrer l'identité des résultats obtenus par ces deux modes d'exposition. *Extract from the paper.*

Crow, James F. A chart of the χ^2 and t distributions. J. Amer. Statist. Assoc. 40, 376 (1 plate) (1945). [MF 13413]

Rubin, Herman. On the distribution of the serial correlation coefficient. Ann. Math. Statistics 16, 211–215 (1945). [MF 12913]

The author proves the equality of approximations given by Koopmans and by Dixon [same Ann. 13, 14–33 (1942); 15, 119–144 (1944); these Rev. 4, 22; 6, 6] to the distribution of the serial correlation coefficient $\hat{r}$, assuming that the true value ρ of $\hat{r}$ is 0, where

$$\hat{r}=\sum_{t=1}^{T}\chi_t\chi_{t+1}\Big/\sum_{t=1}^{T}\chi_t^2, \qquad \chi_{T+1}=\chi_1,$$

and the χ_t are normally and independently distributed with mean 0 and variance σ^2. *R. L. Anderson.*

Thurstone, L. L. A multiple group method of factoring the correlation matrix. Psychometrika 10, 73–78 (1945). [MF 12641]

The author presents a method of reducing an n-square correlation matrix to an $n\times r$ factor matrix by means of the selection of a set of clusters of highly correlated tests. This method avoids the calculation of residuals after the computation of each factor and enables one to extract several factors at the same time. An example is given for $n=9$ with the tests divided into three clusters of three tests each. The selection of constituent parts of the clusters seems to be rather arbitrary, and about the same results can usually be obtained with several different groupings. *R. L. Anderson* (Raleigh, N. C.).

Wald, Abraham. Tests of statistical hypotheses concerning several parameters when the number of observations is large. Trans. Amer. Math. Soc. 54, 426–482 (1943). [MF 9520]

Let $x_1, \cdots, x_r$ be r random variables whose joint probability density function $f(x_1,\cdots,x_r,\theta_1,\cdots,\theta_k)$ is assumed known except for the values of the parameters $\theta_1,\cdots,\theta_k$. The author considers the general problem of testing hypotheses about the values of $\theta_1,\cdots,\theta_k$ when the number of independent observations approaches infinity. A hypothesis H_ω that the point $\theta=\theta_1,\cdots,\theta_k$ lies in a given subset ω of

the parameter space Ω is tested on the basis of n independent observations on $x_1, \cdots, x_r$ as follows. A critical region (subset) W_n of the rn-dimensional sample space is chosen and the hypothesis is rejected if and only if the observed sample point lies in the critical region W_n. Let $P(W_n|\theta)$ denote the probability that the sample point will fall in W_n when θ is the parameter point. The author considers the following three definitions of "best" critical regions. (I) Let F be a given family of surfaces in the parameter space such that each element of F lies entirely in the complement $\bar{\omega}$ of ω. Let $w(\theta)$ be a given nonnegative "weight" function defined over the entire parameter space. A critical region W_n is said to have uniformly best average power with respect to the family F of surfaces and the weight function $w(\theta)$ if

$$\int_{F_0} P(W_n|\theta)w(\theta)dA \geqq \int_{F_0} P(Z_n|\theta)w(\theta)dA$$

for any element F_0 of F and for any region Z_n of the rn-dimensional sample space for which

$$\underset{\theta\epsilon\omega}{\text{l.u.b.}}\ P(Z_n|\theta) = \underset{\theta\epsilon\omega}{\text{l.u.b.}}\ P(W_n|\theta).$$

(II) Let F be as before. A critical region W_n for testing H_ω is said to have uniformly best constant power on the family F if (a) $P(W_n|\theta)$ is constant over each element of F; (b) if Z_n satisfies (a) and the condition

$$\underset{\theta\epsilon\omega}{\text{l.u.b.}}\ P(Z_n|\theta) = \underset{\theta\epsilon\omega}{\text{l.u.b.}}\ P(W_n|\theta),$$

then $P(W_n|\theta) \geqq P(Z_n|\theta)$ for any θ not in ω. (III) Denote by $P_n(\theta, \omega, \alpha)$ the l.u.b. of $P(Z_n|\theta)$ with respect to Z_n subject to the condition that $\text{l.u.b.}_{\theta\epsilon\omega} P(Z_n|\theta) = \alpha$, a preassigned constant. A critical region W_n is said to give a most stringent test of H_ω if, for some positive α, $\text{l.u.b.}_{\theta\epsilon\omega} P(W_n|\theta) = \alpha$ and

$$\underset{\theta\epsilon\bar{\omega}}{\text{l.u.b.}}\ \{P_n(\theta, \omega, \alpha) - P(W_n|\theta)\} \leqq \underset{\theta\epsilon\bar{\omega}}{\text{l.u.b.}}\ \{P_n(\theta, \omega, \alpha) - P(Z_n|\theta)\}$$

for all regions Z_n for which $\text{l.u.b.}_{\theta\epsilon\omega} P(Z_n|\theta) = \alpha$.

One of the main results obtained is the following. Under certain restrictions on the density function f and on H_ω the Neyman-Pearson likelihood ratio test is asymptotically "best" in the sense of each of the three definitions. The family F and weight function $w(\theta)$ with respect to which the likelihood ratio test has asymptotically best average and asymptotically best constant power are of relatively simple structure. The author also derives the limiting form of the power function of the likelihood ratio test when the number of observations approaches infinity. *J. Wolfowitz.*

Wald, Abraham. Statistical decision functions which minimize the maximum risk. Ann. of Math. (2) **46**, 265–280 (1945). [MF 12401]

In the author's theory of statistical inference the problem is, given a weight function of errors, to choose a decision function. The treatment here is parametric and the results extend those of an earlier paper [Ann. Math. Statistics **10**, 299–326 (1939); these Rev. **1**, 152]. Let θ be the true parameter point and ω the decision function; the given weight function then determines a risk function $r(\theta, \omega)$. Various methods of choosing a favorable ω are considered. (1) Minimize $\max_\theta r(\theta, \omega)$. (2) If θ is known to have a distribution $F(\theta)$, minimize $\int r(\theta, \omega)dF(\theta)$, the average risk relative to $F(\theta)$. (3) Minimize the average risk relative to a least favorable $F(\theta)$, where a least favorable $F(\theta)$ is one that maximizes $\min_\omega \int r(\theta, \omega)dF(\theta)$. Interrelations and other properties of these various decision functions are studied. They are related to strategies considered in von Neumann's theory of two-person zero-sum games; the conclusion of the paper restates some of the results in von Neumann's terminology. No applications are given. *H. Scheffé* (Princeton, N. J.).

Wald, A., and Wolfowitz, J. Sampling inspection plans for continuous production which insure a prescribed limit on the outgoing quality. Ann. Math. Statistics **16**, 30–49 (1945). [MF 12356]

Sampling inspection plans are considered for articles produced by a continuous production process. The plans are applicable to articles that can be classified as defective or nondefective and are so designed that the long-run proportion of defectives will not exceed a prescribed limit irrespective of whether the inspection is continuous or "by lots." The concepts of "outgoing quality limit" and "local stability" are discussed. Proofs are given that a described simple inspection plan will insure a prescribed average outgoing quality limit and that this plan requires minimum inspection when the production process is in a state of statistical control. A general class of plans possessing both these properties is also discussed. Some methods for achieving local stability are briefly considered.

W. A. Shewhart (New York, N. Y.).

Scheffé, H., and Tukey, J. W. Non-parametric estimation. I. Validation of order statistics. Ann. Math. Statistics **16**, 187–192 (1945). [MF 12909]

The authors discuss the nonparametric problems of (a) confidence intervals for a quantile; (b) tolerance limits which will cover a given proportion or a greater one of the population. Let $1-\alpha$ be the confidence coefficient. The order statistics of Thompson and Nair for problem (a) and of Wilks for problem (b) are valid when the underlying cumulative distribution function is continuous, without the requirement of the existence of a density function. When all cumulative distribution functions are admitted without restriction the situation may be roughly described as follows. The same confidence intervals or tolerance limits as above correspond to a confidence coefficient at least $1-\alpha$ or at most $1-\alpha$ according as the end points are or are not included. *J. Wolfowitz* (Raleigh, N. C.).

Mann, Henry B. Nonparametric tests against trend. Econometrica **13**, 245–259 (1945). [MF 12786]

The author deals with the problem of testing the null hypothesis that a sequence of variates $X_1, \cdots, X_n$ are independently distributed, each with the same distribution, against the alternative hypothesis that there is a downward trend. The sequence $X_1, \cdots, X_n$ is said to have a downward trend if $X_1, \cdots, X_n$ are independently distributed with cumulative distribution functions $f_1(X), \cdots, f_n(X)$, respectively, such that $f_i(X) < f_{i+k}(X)$ for every i, every X and every positive k. The author proposes two tests against downward trend, the T-test and K-test, and gives sufficient conditions for their consistency and unbiasedness.

The T-test is defined as follows. Let T denote the number of pairs $(k, l)(k, l = 1, \cdots, n; k < l)$ for which $X_k < X_l$ in the observed sample. The null hypothesis is rejected if and only if T does not exceed a properly chosen constant $\bar{T}$. Certain alternatives are discussed against which the T-test is most powerful. The statistic T, as pointed out by the author, was first proposed by M. G. Kendall for testing independence in a bivariate distribution.

If the alternative to the null hypothesis is likely to be such that the probability that $X_i > X_j$ increases rapidly with $j-i$, the K-test is suggested as being more powerful than the T-test. The K-test is defined as follows. Let $X_0, \cdots, X_{n-1}$ be the sample; K is defined as the smallest positive integer with the property that $X_i > X_j$ holds for all pairs (i, j) $(i=0, \cdots, n-k-1; j=k, \cdots, n-1)$ for which $j-i \geqq k$. The null hypothesis is rejected if and only if K does not exceed a properly chosen constant $\bar{K}$. The author discusses the determination of the probability that $K \leqq \bar{K}$ under the null hypothesis. *A. Wald* (New York, N. Y.).

Mann, Henry B. On a test for randomness based on signs of differences. Ann. Math. Statistics 16, 193–199 (1945). [MF 12910]

The test is one proposed by Moore and Wallis [J. Amer. Statist. Assoc. 38, 153–164 (1943); these Rev. 4, 281]. Let the sample values in order of observation be $x_1, \cdots, x_n$ and let S be the number of negative differences $x_i - x_{i-1}$, $i=0, 1, \cdots, n$. If the sample all comes from the same universe, $p_i = p(x_i > x_{i+1}) = \frac{1}{2}$; in this case the author finds a recurrence formula for the moments of S and the limiting form of the standard moments of S for $n \to \infty$. From the latter result it follows by standard methods that the distribution of S tends to normality with large n, confirming empirical findings of Moore and Wallis. The power of this test is studied with respect to the class of alternate hypotheses given by $p_i = \frac{1}{2} + \epsilon_i$, $\sum_{i=1}^{n} \epsilon_i = \lambda_n(n-1)$, $\liminf_{n\to\infty} \lambda_n = \lambda > 0$, which may frequently hold in manufacturing processes. A lower bound which depends only on λ_n is found for this power for large n. *C. C. Craig* (Ann Arbor, Mich.).

Kaitz, Hyman B. A note on reliability. Psychometrika 10, 127–131 (1945). [MF 12643]

When only the distribution of total scores on a test is available, the Kuder-Richardson formula for reliability [Psychometrika 2, 151–160 (1937)] was adapted by its originators to give an approximation to the reliability in the case test items are weighted one or zero. In the present paper, adapting the development by the means of the analysis of variance used by Hoyt [Psychometrika 6, 153–160 (1941)], the author gives a formula for reliability in the case that only total scores are available and test items may have weights other than zero or one. *C. C. Craig.*

Barricelli, Nils Aall. Les plus grands et les plus petits maxima ou minima annuels d'une variable climatique. Arch. Math. Naturvid. 46, no. 6, 155–194 (1943). [MF 12981]

Gumbel's method of extreme values is applied to certain empirical meteorological time series. Formal modifications are introduced for the case where the distribution functions vary from year to year. *W. Feller* (Ithaca, N. Y.).

Delgleize, A. Sur la fonction de Makehan. Bull. Soc. Roy. Sci. Liége 11, 163–166 (1942). [MF 13098]

The author suggests a slight modification of G. King's and G. F. Hardy's well-known method for graduating mortality tables according to Makeham's formula. Instead of dividing the number living into four groups of t ages, each group corresponding to the age interval $[a+jt, a+(j+1)t-1]$ for $j=0, 1, 2, 3$, the author proposes to form four groups of t ages, where the jth group contains ages $a+j+4k$ with $k=0, 1, 2, \cdots, t-1$; $j=0, 1, 2, 3$. *E. Lukacs.*

GEOMETRY

Lalan, V. Sur certaines équations fonctionnelles et les fondements de la géométrie. Bull. Soc. Math. France 72, 55–67 (1944). [MF 13218]

If two orthogonal vectors OV_1 and OV_2 are together equivalent to a single vector OV making an angle α with OV_1, we have $OV_1 = OV \cos\alpha$, $OV_2 = OV \sin\alpha$. This familiar result in the Euclidean plane remains valid in the non-Euclidean planes, although of course OV_1 and OV_2 will no longer be orthogonal projections of OV. The vectors represent "virtual velocities" for a continuous movement of the plane on itself. Thus, for translation along a line with velocity V_0, the velocity of a point distant a from this line is $V_0\varphi(a)$, where $\varphi(0)=1$ and $\varphi(a+b)+\varphi(a-b)=2\varphi(a)\varphi(b)$. It follows that $\varphi(a)=1$ (Euclidean case) or $\cos(a/k)$ (elliptic case) or $\cosh(a/k)$ (hyperbolic case). Such considerations lead quite simply to the theory of circles, horocycles, hypercycles (or equidistant curves) and to the fundamental formulas of non-Euclidean trigonometry and differential geometry. *H. S. M. Coxeter* (Toronto, Ont.).

Momet, Pierre. Sur le théorème fondamental de la géométrie projective. Revue Sci. (Rev. Rose Illus.) 79, 140–146 (1941). [MF 12858]

The main object of this paper is to give a proof of the so-called fundamental theorem of projective geometry which does not make any use of Dedekind's axiom of continuity and is based mainly on the axiom of Archimedes. To do this one has to show that the theorem of Pappus is a consequence of the axiom of Archimedes. This is well known to be equivalent to proving that a field is commutative if it admits an Archimedean algebraic ordering; this last fact is well known. The axioms are stated rather loosely, so much so that it may prove necessary to incorporate some of the propositions into the axioms when giving a complete and precise account. *R. Baer* (Urbana, Ill.).

Popoviciu, Tiberiu. Quelques remarques sur un théorème de M. Pompeiu. Bull. Math. Soc. Roumaine Sci. 43, 27–43 (1941). [MF 12727]

Let $A_1, \cdots, A_n$ be the vertices of a regular n-gon, P any point in its plane. The author proves that

$$n^{-1}\{\overline{PA}_1 + \cdots + \overline{PA}_n\} \geqq C_n^{(1)} \max \overline{PA}_i,$$

where $C_n^{(1)} = n^{-1} \cot(\frac{1}{2}\pi/n)$ for even n and $n^{-1} \csc(\frac{1}{2}\pi/n)$ for odd n. Pompeiu had proved this result previously for $n=3$. The author also proves several other theorems. For example, if A_1, A_2, A_3, A_4 are the vertices of a square and P is any point, then we can always construct a polygon from sides whose lengths equal $\overline{PA}_i^2$. The square is the only quadrilateral with this property. *P. Erdös.*

Nestorovitch, N. M. Sur la puissance constructive d'un complexe E sur le plan de Lobatchevski. C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 186–188 (1944). [MF 11616]

It is found that problems of the second degree concerning the plane of Bolyai-Lobachevsky can be solved by ruler and compass. *J. L. Dorroh* (Baton Rouge, La.).

Gambier, Bertrand. Triangles en position isogonale. Bull. Soc. Math. France 70, 31–39 (1942). [MF 11887]

By means of any triangle $A_1A_2A_3$ a transformation in the plane is established as follows. To each point M there corre-

sponds as image the point M' which has the property that the line A_iM' is the reflection in each of the bisectors of the angles at A_i of A_iM ($i=1, 2, 3$). The properties of this transformation are developed with emphasis on applications to the theory of Simpson lines. *J. L. Dorroh.*

Labra, Manuel. Calculation of the sides of important copedal triangles. Revista Soc. Cubana Ci. Fís. Mat. 1, 177–187 (1944). (Spanish) [MF 11941]

The lengths of the sides of a Miquel triangle of a point are computed. In addition to the general case, special treatment is given those cases in which the point is the baricenter, orthocenter, incenter, Brocard point or symmedian point of the triangle of reference. *J. L. Dorroh.*

Lu, Chin-Shih. Some new properties of the triangle. Nat. Math. Mag. 19, 398–405 (1945). [MF 13288]

We have dealt with twenty-four points all of which are obtained by drawing tangents to the nine-point circle at these nine points. *Extract from the paper.*

Thébault, V. Sur le point de Kantor-Hervey. Bull. Soc. Roy. Sci. Liége 13, 293–296 (1944). [MF 13183]

Thébault, V. Sur le tranchet d'Archimède. Bull. Soc. Math. France 72, 68–75 (1944). [MF 13219]

Thébault, V. Sur les sphères de Lemoine du tétraèdre. Bull. Soc. Math. France 71, 67–77 (1943). [MF 13233]

Narasinga Rao, A. On the metric geometry of a cyclic n-point. I. Math. Student 12, 91–97 (1945). [MF 12492]

The author considers n points lying on a circle of center T and unit radius. The line TG joining T to the centroid G of the given points is defined as the Euler line of the n points and the point H of this line such that $TG:TH=1:n$ is the orthocenter of the n points. The midpoint M of TH is the medial point, the circle of center M and radius equal to $\frac{1}{2}$ is the medial circle and the perpendicular at M to TH is the central line of the n points. These elements enable the author to establish a number of propositions which for the case $n=3$ are identical with the classical properties of the orthocenter, the nine-point circle and the polar circle of a triangle. The proofs are based on the use of complex coordinates. *N. A. Court* (Norman, Okla.).

Narasinga Rao, A., and Venkataraman, M. On the Clifford and Grace chains. Math. Student 12, 98–101 (1945). [MF 12493]

An expository article.

Rangachariar, V., and Singh, B. On the radical conic of two central conics. Math. Student 12, 86–87 (1945). [MF 12490]

Kesava Menon, P. An extension of a theorem of Steiner. Math. Student 12, 78–79 (1945). [MF 12487]

The proposed extension is as follows. If two sets of three circles a, b, c; a', b', c' be such that the three circles coaxal with b, c; c, a; a, b and orthogonal to a'; b'; c' respectively are coaxal, then the three circles coaxal with b', c'; c', a'; a', b' and orthogonal to a; b; c respectively are also coaxal. *Extract from the paper.*

Diaz, José Gallego. On a complex projectivity related to a given conic. Gaz. Mat., Lisboa 4, no. 13, 1–2 (1943). (Portuguese) [MF 12961]

We establish a simple complex projectivity in the plane of a conic, such that its double points are the foci of the conic. *Extract from the paper.*

Olmsted, John M. H. Matrices and quadric surfaces. Nat. Math. Mag. 19, 267–275 (1945). [MF 12474]

The author classifies quadric surfaces by means of the ranks of two matrices and the signs of their characteristic roots. [The results are essentially included in a paper of R. S. Burington, Amer. Math. Monthly 39, 527–532 (1932).] *C. C. MacDuffee* (Madison, Wis.).

Abellanas, Pedro F. Formulas for the Cremona characteristics of complete quadrics. Revista Mat. Hisp.-Amer. (4) 4, 3–9 (1944). (Spanish) [MF 12158]

The symbol of Schubert for a tetradimensional system of quadrics is expressed in terms of the symbols for three associated systems. *J. L. Dorroh* (Baton Rouge, La.).

Jongmans, F. Sur les mouvements d'un espace à quatre dimensions. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 650–665 (1941). [MF 12677]

Let m and n be two fixed lines in projective 3-space. From any point P we can draw a unique transversal PMN to these lines. Let P' be the harmonic conjugate of P with respect to M and N. The correspondence $P\rightarrow P'$ is called a harmonic biaxial homography: hyperbolic if m and n are real, elliptic if they are conjugate complex. There is a linear congruence of invariant lines (namely, the transversals of m and n) on each of which the homography induces an involution. Let Ω_0 be an elliptic harmonic biaxial homography, transforming a real line p into p'. Then the hyperbolic harmonic biaxial homography with axes p and p' commutes with Ω_0. What other collineations will commute with Ω_0? The paper answers this question by considering in turn all possible types of collineation. The conclusion is that every such collineation is a product of not more than nine hyperbolic harmonic biaxial homographies of the above kind. *H. S. M. Coxeter* (Toronto, Ont.).

Araujo, Roberto. Application of the associated elements of a homography to the study of the anallagmatic conics. Revista Mat. Hisp.-Amer. (4) 3, 377–386 (1943). (Spanish) [MF 12180]

The term "anallagmatic" is used not in the customary sense of "invariant under inversion," but as a synonym of the term "invariant." The author obtains a necessary and sufficient condition for a nonhomological homography to admit invariant conics. *N. A. Court* (Norman, Okla.).

Rodeja F., E. G-. Anallagmatic conics of a plane homography. Revista Mat. Hisp.-Amer. (4) 4, 149–152 (1944). (Spanish) [MF 12169]

The ground covered is the same as in the paper reviewed above. The author uses a different approach and makes no mention of that paper. *N. A. Court* (Norman, Okla.).

Godeaux, Lucien. Remarques sur les homographies cycliques du plan. Bull. Soc. Roy. Sci. Liége 9, 137–143 (1940). [MF 13041]

Godeaux, Lucien. Sur les points unis des homographies cycliques de l'espace. Bull. Soc. Roy. Sci. Liége 10, 15–22 (1941). [MF 13048]

Godeaux, Lucien. Sur un groupe d'homographies planes. Bull. Soc. Roy. Sci. Liége 10, 23–32 (1941). [MF 13049]

Godeaux, Lucien. Sur une homographie hyperspatiale de période quatre. Bull. Soc. Roy. Sci. Liége 10, 214–218 (1941). [MF 13060]

Calvo, D. Sur les réciprocités de l'espace dont les homographies associées n'ont que deux droites unies. Bull. Soc. Roy. Sci. Liége 10, 67–72 (1941). [MF 13052]

Calvo, D. Sur les réciprocités réelles de l'espace dont les homographies associées sont dépourvues de points réels unis. Bull. Soc. Roy. Sci. Liége 10, 301–306 (1941). [MF 13068]

Calvo, Dolorès. Remarque sur les réciprocités du plan et de l'espace. Bull. Soc. Roy. Sci. Liége 10, 463–465 (1941). [MF 13075]

Legrand, M. Sur les homographies de l'espace n'ayant que deux droites unies. Bull. Soc. Roy. Sci. Liége 10, 73–76 (1941). [MF 13053]

Legrand, M. Sur les homographies de l'espace n'ayant qu'une droite unie. Bull. Soc. Roy. Sci. Liége 10, 186–191 (1941). [MF 13059]

Bosquet, René. Sur les homographies permutables avec une polarité uniforme du plan. Bull. Soc. Roy. Sci. Liége 12, 218–220 (1943). [MF 13139]

Lorent, H. Une transformation des surfaces et des lignes de l'espace. Bull. Soc. Roy. Sci. Liége 10, 562–573 (1941). [MF 13083]

Lorent, H. La transformation par hyperbolisme étendue à l'espace. Bull. Soc. Roy. Sci. Liége 11, 523–528 (1942). [MF 13121]

Rozet, O. Note de géométrie projective. Bull. Soc. Roy. Sci. Liége 10, 460–462 (1941). [MF 13074]

Lagrange, René. Définitions et théorèmes de métrique anallagmatique. Ann. Sci. École Norm. Sup. (3) 59, 1–42 (1942). [MF 11878]

In anallagmatic geometry, in addition to the sphere, the elements are the point, the circle, and the point couple, or bipoint, and are to be regarded as the intersection of the members of a single or double linear family of spheres. For each type of element pair (not both points) a numerical function called distance is defined. If one of the elements of a pair is a point, this distance is a covariant under anallagmatic transformations. If neither of the elements of a pair is a point, the distance is an invariant. A cocyclic field of bipoints is a set of bipoints each two of which are cocyclic. For any cocyclic field of bipoints there is a sphere g, real or imaginary, such that each bipoint in the field consists of a pair of points mutually inverse in g. Models for elliptic geometry and for hyperbolic geometry are given by cocyclic fields. *J. L. Dorroh* (Baton Rouge, La.).

Bouligand, Georges, et Choquet, Gustave. Problèmes liés à des métriques variationnelles. C. R. Acad. Sci. Paris 218, 696–698 (1944). [MF 12715]

For a three-dimensional generalized Finsler space, definitions are given for area of a two-dimensional submanifold, flux through a surface element, generalized Laplacian, directional derivative and minimal surface. Some problems, but no theorems, are formulated. *H. Busemann.*

de Mira Fernandes, Aureliano. Distance geometry, I. Generalities—vector algebra. Cadernos de Análise Geral, no. 13. Junta de Investigação Matemática, Pôrto, 1945. 16 pp. (Portuguese) [MF 12936]

This article reproduces [without explicit reference to it] a small part of a paper by Menger [Enseignement Math. 35, 348–372 (1936)]. Two typographical errors which occur in formulae on pages 354 and 360 of the earlier paper are preserved in the present one [p. 3, line 9; p. 8, line 9]. *L. M. Blumenthal* (Columbia, Mo.).

Anadon Laplaza, Santos. The conformal cartography of the ellipsoid of revolution with applications to the map of Spain. II. Revista Mat. Hisp.-Amer. (4) 1, 141–162, 215–228 (1941). (Spanish) [MF 12995]

Part I appeared in the same Revista (3) 2, 8–43 (1940); these Rev. 3, 252; 6, 334.

Zagrebin, D. Stokes formula for the case of an ellipsoidal level surface. Bull. Inst. Astr. Acad. Sci. URSS no. 52, 407–435 (1944). (Russian. English summary) [MF 12890]

The present paper gives an integral expression for the undulations of a geoid relative to an ellipsoidal level surface. The expression in question may be considered as a development of Stokes' classical formula. *Author's summary.*

Convex Domains, Integral Geometry

Lagrange, René. Sur le tétraèdre et sur la sphère minimum contenant un ensemble de points. Bull. Sci. Math. (2) 67, 108–115 (1943). [MF 12634]

Let there be given a set of diameter 1 in n-dimensional space. The author proves that for $n=2$ or 3 there always exists a sphere of diameter at most $\{2n/(n+1)\}^{\frac{1}{2}}$ containing the set. [This result has been proved for general n by Jung [J. Reine Angew. Math. 123, 241–257 (1901)].] *P. Erdös* (Stanford University, Calif.).

Hadwiger, H. Über extremale Punktverteilungen in ebenen Gebieten. Math. Z. 49, 370–373 (1944). [MF 11999]

Let G be a connected (closed) domain whose Jordan area is F and whose boundary has the finite length L. Let $d(V_n)$ denote the smallest distance between any two points of a set V_n of n points on G, and let d_n be the least upper bound of $d(V_n)$ if V_n runs through all such sets ($n>2$, fixed). Then $d_n^2 \geqq 2F/(n\sqrt{3})$. A more complicated upper bound is derived for d_n, depending not only on F and n but also on L. As a corollary, the author obtains the following formula of Fejes [Math. Z. 46, 83–85 (1940); these Rev. 1, 263]:

$$\lim_{n\to\infty} nd_n^2 = 2F/\sqrt{3}.$$

P. Scherk (Saskatoon, Sask.).

Hadwiger, H. Eine elementare Ableitung der isoperimetrischen Ungleichung für Polygone. Comment. Math. Helv. 16, 305–309 (1944).

The author gives a completely elementary proof of the sharpened isoperimetric inequality for convex polygons P:

$$L^2 - 4\pi F \geqq (\pi^2/4)(s-2r)^2.$$

Here L is the length of P, F its area, r the radius of the largest inscribed circle and s the length of any chord through the center of a largest inscribed circle. For any polygon Q (not necessarily convex) the parallel region $Q(\rho)$ at distance ρ is defined to be that region whose points are at distances not exceeding ρ from P. If $F(\rho)$ is the area of $Q(\rho)$, an inequality of Steiner states that $F(\rho)\leqq F+L\rho+\pi\rho^2$. The author proves this inequality by induction on the number of sides of Q. The theorem itself is proved by enclosing P in a large square and applying Steiner's inequality to the two polygons into which the ring-shaped figure outside P and inside the square is divided by an arbitrary line through the center of the largest circle inscribable in P.

J. W. Green (Los Angeles, Calif.).

Adler, Claire Fisher. An isoperimetric problem with an inequality. Amer. Math. Monthly 52, 59–69 (1945). [MF 11903]

From known results about the classical isoperimetric problem the author derives, by elementary arguments, the following generalization. A plane curve C is "admissible" if (i) it is closed, continuous, and sectionally smooth; (ii) any point in the rest of the plane has index 0 or 1 with respect to C, and is said to belong to its exterior or interior accordingly; (iii) C and its interior contain a given finite set of points; (iv) the interior of C has a given area $S\geqq 0$. Problem: to find an admissible curve of minimum length. It is shown that there is at least one solution, any solution consisting, in general, of circular arcs (all convex or all concave) with a common radius and of straight segments (each described twice); if $S=0$ there are no arcs; if S is sufficiently large, no segments.

H. P. Mulholland (Beirut).

Dinghas, Alexander. Über die isoperimetrische Eigenschaft der Kugel im gewöhnlichen Raum. Monatsh. Math. Phys. 51, 153–172 (1944). [MF 12481]

The author gives a proof of the isoperimetric property of the sphere in the class of "regular bodies of E. Schmidt." These bodies are not necessarily convex, but are locally parametrizable by twice differentiable functions. According to the author, the virtue of this proof is that no use is made of symmetrization. The proof begins with a sharp form of the Brunn-Minkowski theorem for a body consisting of a system of polyhedra. The method used is essentially that of E. Schmidt. The Brunn-Minkowski inequality leads immediately to a sharp isoperimetric inequality for polyhedra. Finally, any "regular body" P can be approximated by polyhedra P_n whose volumes and areas converge to those of P, when area is computed by the classical integral formula.

J. W. Green (Los Angeles, Calif.).

Dinghas, Alexander. Über eine isoperimetrische Aufgabe von Erhard Schmidt. I. Math. Z. 49, 734–792 (1944). [MF 11979]

This paper belongs to a series by the author and E. Schmidt on isoperimetric problems in spaces of constant curvature [see Dinghas, Math. Ann. 118, 636–686 (1943); these Rev. 6, 101]. Using the same notation as in the review cited, the problem may be formulated as follows. Determine, in n-dimensional Cayley-Klein space, among all "admissible" solids of given surface contained in the ring-shaped region

$$a^{-1}\tan^2 a\sqrt{K}\leqq\frac{u_1^2+\cdots+u_{n-1}^2}{1+Ku_n^2}\leqq b^{-1}\tan^2 b\sqrt{K},$$

the one of greatest volume. The problem is solved in the following steps.

(1) Determination of the extremals in the isoperimetric two-dimensional problem to which the n-dimensional problem reduces for solids of revolution. Rotation of the extremal curves about a coordinate axis leads to "extremal solids," of which three different types are distinguished.

(2) Proof of the "isoperimetric identity"

$$(n-1)(V-V_g)=\rho_g(O-O_g)-2\rho_g\int\sin^2\tfrac{1}{2}(\bar{n},\bar{\bar{n}})dO.$$

Here V and O are volume and surface of an "admissible" solid k, V_g and O_g are volume and surface of an extremal solid k_g, ρ_g is a constant depending on the parameter g, and $\bar{n}$ and $\bar{\bar{n}}$ are the normals of k and k_g at points between which a certain correspondence has been established. In the proof of this identity essential use is made of notions developed by E. Schmidt [Math. Z. 44, 689–788 (1939)] in the proof of Gauss's divergence theorem.

(3) Proof that (for a certain g) there is an extremal solid of one of the three types with $O=O_g$. The identity then furnishes the inequality $V\leqq V_g$.

F. John.

Dinghas, Alexander. Über einen geometrischen Satz von Wulff für die Gleichgewichtsform von Kristallen. Z. Kristallogr., Mineral. Petrogr. Abt. A. 105, 304–314 (1944). [MF 11810]

Let P be a crystal and F_i, $i=1,\cdots,n$, its face areas; σ_i is a positive constant associated with F_i and called the free energy per unit area of F_i. A theorem of Gibbs and Curie states that, for the equilibrium shape of the crystal, the total free energy $\Phi=\sum F_i\sigma_i$ is a minimum among crystals P of fixed volume with at most n faces. The theorem of Wulff states that this minimum Φ_r is attained by a crystal P_r containing a point O such that the distances from O to the faces F_i are proportional to σ_i. The author gives a new proof of Wulff's theorem and proves that, for every other competing crystal, $\Phi>\Phi_r$. If $\{\bar{N}_i\}$ is a set of n unit vectors emerging from the origin O and satisfying suitable linear independence conditions, and $\{r_i\}$ is a set of positive numbers, then the set of points Q such that $\overline{OQ}\cdot\bar{N}_i\leqq r_i$, $i=1,\cdots,n$, is a convex polyhedron P_r of n or fewer faces; the exterior normals to the faces are among the $\{\bar{N}_i\}$ and the distance from O to the face whose normal is $\bar{N}_i$ is r_i. Let $P_{\lambda\sigma}$ be that P_r obtained by putting $r_i=\lambda\sigma_i$, where λ is a positive constant; $P'_{\lambda\sigma}$ is the convex body obtained by placing the center of $P_{\lambda\sigma}$ at every point of P_r and taking the total volume covered in this way. If V stands for volume, then the inequality $V'^{\frac{1}{3}}_{\lambda\sigma}\geqq V_r^{\frac{1}{3}}+V_{\lambda\sigma}^{\frac{1}{3}}$ is obtained. This is a special case of the Brunn-Minkowski inequality; the proof used is a simple one due to E. Schmidt. Combining this inequality with the geometrically deduced fact that

$$\lim_{\lambda\to 0}\lambda^{-1}(V'_{\lambda\sigma}-V_r)=\sum F_i\sigma_i=\Phi,$$

the inequality $\Phi\geqq 3V_r^{\frac{2}{3}}V_\sigma^{\frac{1}{3}}$ is obtained, and it is shown that the equality can hold only when P_r is a $P_{\lambda\sigma}$. The stated theorem follows immediately.

J. W. Green.

Santaló, Luis A. Area bounded by the curve generated by the end of a segment whose other end traces a fixed curve, and application to the derivation of some theorems on ovals. Math. Notae 4, 213–226 (1944). (Spanish) [MF 12206]

Let $\mathfrak{x}=\mathfrak{x}(s)$ be the parametric representation of a closed curve C (s, the arc length). Let L be the length and F the

area of C. Let $\mathfrak{y}(s)=\mathfrak{x}(s)+h(s)\mathfrak{e}(s)$ be the representation of another closed curve C' ($\mathfrak{x}'^2=1$; $h(s)>0$). Then the area of C' is equal to

$$\Phi=F+\oint h\mathfrak{e}\times\mathfrak{x}'ds+\tfrac{1}{2}\oint h^2\mathfrak{e}\times\mathfrak{e}'ds. \tag{1}$$

Let $\rho(s)$ denote the radius of curvature of C and let $\theta=\sin^{-1}\mathfrak{e}\times\mathfrak{x}'$ be the angle between the tangent vector $\mathfrak{x}'$ and $\mathfrak{e}$. The author studies the following cases.

(I) $C'=C$; $\theta=$constant or $h=$constant. Then from (1), $h=2\rho\sin\theta$ for at least two points of C. If C is convex the existence of at least four such points is proved. If both θ and h are constant, (1) yields $L=\pi h/\sin\theta$; $\theta=\pi/2$ yields the special case of the curves of constant width. (II) $\theta=\pi/2$, $h=$constant; C is convex and C' is an inner parallel curve of C. Then (1) becomes

$$\Phi=F-hL+h^2\pi. \tag{2}$$

Let $\rho_M=\max_s\rho(s)$, $\rho_m=\min_s\rho(s)$. If $h>\rho_M$, the domain bounded by C' can be characterized as the locus of the centers of those circles of radius h which contain C. If $h\leqq\rho_m$ or $h\geqq\rho_M$, C' is convex; in particular, $\Phi\geqq 0$. In connection with (2) this yields upper bounds for the isoperimetric deficit. Lower bounds are obtained by choosing the coordinate system in a suitable way, estimating the right side of (1) directly, and thus proving Bonnesen's inequality that $\Phi\leqq 0$ if $2h$ is equal to a width of C [Bonnesen, Les Problèmes des Isopérimètres et des Isépiphanes, Gauthier-Villars, Paris, 1929, p. 61].

The proof of Bonnesen's inequality is not quite clear to the reviewer. For, the author's formula (6.3) applies to ordinary Cartesian coordinates, while the axes he has constructed do not seem necessarily rectangular. However, they are sure to be so if $2h$ is equal to a largest or a smallest width of C. Thus $\Phi\leqq 0$ for these two values of h; Φ being a quadratic polynomial in h and positive for large h, we may conclude that $\Phi<0$ for all h between these two values; that is, $\Phi\leqq 0$ if $2h$ is equal to any width of C. *P. Scherk.*

Santaló, L. A. Note on convex curves on the hyperbolic plane. Bull. Amer. Math. Soc. 51, 405–412 (1945). [MF 12520]

Let C be an oriented convex curve in the hyperbolic plane of curvature -1. Thus C has at most two points in common with any straight line. Let L be the length of C, F the area of C. If a straight line of support g_O at a point O on C is given, then any point P on C can be determined by the length s of the arc OP or by the angle τ between g_O and a straight line of support g_P of C at P. Let h_P denote the perpendicular to g_P at P. If a second line of support perpendicular to h_P intersects h_P at P', let $\alpha=PP'$. We assume that C is composed of a finite number of arcs each with a continuous curvature $\kappa>1$. If g is an oriented straight line which intersects C, then there exists one and only one h_P perpendicular to g such that P lies to the right of g. Let a denote the distance between P and g; thus $0<a<\alpha$. The author obtains the following formula for the measure of the g's:

$$dg=\cosh a\,da\,d\tau-\sinh a\,da\,ds. \tag{1}$$

Since $2L=\int_{C\cdot g\neq 0}dg$, (1) yields the "principal formula"

$$L=\oint\sinh\alpha\,d\tau-\oint\cosh\alpha\,ds. \tag{2}$$

Various applications of (2) are given. In particular, the curves of constant breadth α are studied; for them, (2) implies $L=(2\pi+F)\tanh\frac{1}{2}\alpha$.

P. Scherk (Saskatoon, Sask.).

Santaló, L. A. Mean value of the number of regions into which a body in space is divided by n arbitrary planes. Revista Unión Mat. Argentina 10, 101–108 (1945). (Spanish) [MF 12504]

Let the body K be a topological image of the sphere in Euclidean 3-space. Let V be the volume of K, F the area of K, M the integral of the mean curvature of K, M_0 the integral of the mean curvature of the convex cover of K. Thus $M_0=\int dE$, where the differential dE is the density of the planes E in space and the integration is extended over all the E that intersect K. Let $E_1,\cdots,E_n$ be n arbitrary planes. Let R be the number of regions into which they divide K, V_i the number of points of intersection of any three planes inside of K, V_e the number of points of intersection of any two planes with K; let $\bar{R}$, $\bar{V}_i$, $\bar{V}_e$ denote the corresponding mean values if the n planes are moved in space. Thus $\bar{R}=M_0^{-n}\int RdE_1\cdots dE_n$ is the mean value of the number of regions into which K is divided by n arbitrary planes. Using Euler's formula, the author proves that in general

$$R=V_i+\tfrac{1}{2}V_e+\sum_{h=1}^{n}\nu_h+1,$$

where ν_h is the number of closed contours into which the intersection of K with E_h is decomposed. Well-known formulas of integral geometry readily yield

$$\bar{V}_i=\binom{n}{3}\pi^4VM_0^{-3},\quad \bar{V}_e=\binom{n}{2}\pi^3F2^{-1}M_0^{-2},\quad \int\nu_h dE=M.$$

Hence $\bar{R}=\binom{n}{3}\pi^4VM_0^{-3}+\binom{n}{2}\pi^3F(2M_0)^{-2}+nM_0^{-1}M+1$.

P. Scherk (Saskatoon, Sask.).

Algebraic Geometry

Chevalley, Claude. Intersections of algebraic and algebroid varieties. Trans. Amer. Math. Soc. 57, 1–85 (1945). [MF 11910]

The author develops a theory of intersections of algebraic varieties which is more general than the theory of van der Waerden [Math. Ann. 115, 619–642 (1938)]. It is based on a study of the local properties of varieties in the neighborhood of their intersections and has the advantage over van der Waerden's theory that, while the latter only attributes a multiplicity to a component of the intersection of two varieties U and V, of dimensions u and v, respectively, lying in a space of n dimensions, when each component of the intersection is of dimension $u+v-n$, the present theory attributes a multiplicity to any component of this dimension, irrespective of the dimension of the other components of the intersection.

Part I develops properties of geometric local rings, which are fundamental in the theory. The most important results are: (i) a theorem of transition, which enables us to reduce problems on the intersections of algebraic varieties to similar problems for algebroid varieties; (ii) a formula of associativity from which the associativity formula of intersections is derived.

Part II deals with the intersection theory of algebroid varieties. An algebroid variety U is defined in an n-dimensional local space $E^n(X_1,\cdots,X_n)$ over a closed field K by

an ideal $\mathfrak{u}$ in the ring $K[[X_1, \cdots, X_n]]$ of formal power series in $X_1, \cdots, X_n$. The neighborhood ring $\mathfrak{R}(U)$ of U is defined as the ring of quotients of $\mathfrak{u}$ with respect to $K[[X_1, \cdots, X_n]]$; if U is a subvariety of V, $\mathfrak{R}(V)$ is the ring of quotients with respect to $\mathfrak{R}(U)$ of an ideal $\mathfrak{v}$ and the factor ring $\mathfrak{R}_V(U)=\lambda$ is the neighborhood ring of U with respect to V. If, now, U and V are two algebroid varieties of dimensions u and v, respectively, and M is a component of the intersection of dimension $u+v-n$, the multiplicity $i(M; U\cdot V)$ of M is defined as follows. A copy V' of V is taken in a space $E^n(X_1', \cdots, X_n')$ and the product $U\times V'$ formed. Corresponding to M there is a variety M^Δ on $U\times V'$ and on the diagonal $X_i=X_i'$ in $E^n(X)\times E^n(X')$; $i(M; U\cdot V)$ is defined as the multiplicity of the ring $\mathfrak{R}_{U\times V'}(M^\Delta)$ with respect to $x_1-x_1', \cdots, x_n-x_n'$ [cf. Ann. of Math. (2) **44**, 690–708 (1943); these Rev. **5**, 171], where x_i, x_i' are the functions induced in $U\times V'$ by X_i, X_i'. From this a theory of intersections which has all the usual properties is developed.

In part III the results of part II are applied to develop an intersection theory of algebraic varieties. Intersections of algebraic varieties are defined in a manner similar to the definition for algebroid varieties. An application of the transition theorem enables us to replace, locally, an algebraic variety of dimension u by a number of sheets, which are algebroid varieties of dimension u. If, now, M is a component of dimension $u+v-n$ of the intersection of two varieties U and V, of dimensions u and v, respectively, and $\bar{M}$ is any sheet of M it is shown that the multiplicity of M in the intersection of U and V is equal to the sum of the multiplicities of $\bar{M}$ as an intersection of a sheet of U and a sheet of V, over all the sheets of U and V. This enables us to deduce many properties of the intersections of algebraic varieties from those of algebroid varieties. The theory which results is a completely satisfactory theory of intersections.

W. V. D. Hodge (Cambridge, England).

Hodge, W. V. D. On multiple curves. I. Proc. Cambridge Philos. Soc. **41**, 111–117 (1945). [MF 12842]

If a variety Γ contains an unramified involution I_n, of order n, generated by an Abelian group $\mathfrak{G}$, of order n, of birational transformations of Γ into itself, Γ is said to be a normal multiple of the variety C which represents the sets of I_n. If C and Γ are curves, every 1-cycle of the Riemann surface of C corresponds to a transformation of $\mathfrak{G}$, and this gives a homomorphism of the 1-dimensional homology group of C onto $\mathfrak{G}$, which can thus have at most $2p$ generators (p being the genus of C). A Riemann surface for Γ is constructed as follows [reviewer's notation]. Copies $C_1, \cdots, C_n$ of C are taken corresponding to the elements $g_1, \cdots, g_n$ of $\mathfrak{G}$ and each C_k is cut along the retrosections a_k^i, b_k^i ($i=1, \cdots, p$) corresponding to the retrosections a^i, b^i of C. The left banks of a_k^i and b_k^i are then joined with the right banks of a_l^i and b_m^i, where $g_l=g_k\beta_i$, $g_m=g_k\alpha_i$, and α_i and β_i are the elements of $\mathfrak{G}$ corresponding to the cycles a^i and b^i of C. If $\mathfrak{G}$ has fewer than p generators the construction can be simplified, and if $\mathfrak{G}$ is cyclic the Riemann surface of Γ is homeomorphic with a ring of n $(p-1)$-handled spheres joined in series by n interconnecting tubes.

D. B. Scott (London).

Hodge, W. V. D. On multiple curves. II. Proc. Cambridge Philos. Soc. **41**, 117–126 (1945). [MF 12843]

The Abelian group $\mathfrak{G}$ of the paper reviewed above is the direct product of g cyclic groups $\mathfrak{G}_i$ generated by elements σ_i of order $q_i^{n_i}$ (q_i a prime) and it is possible to find a base $u_1, \cdots, u_\pi$ (where $\pi=n(p-1)+1$ is the genus of Γ) for the integrals of the first kind on Γ such that, for an arbitrary element $\sigma=\prod_{i=1}^g\sigma_i^{k_i}$ $(0\leqq k_i<n_i)$ of $\mathfrak{G}$ and an arbitrary point ξ of Γ,

$$(1)\qquad u_j(\sigma\xi)=\left\{\prod_{i=1}^{g}(\alpha_{ij})^{k_i}\right\}u_j(\xi)+c,\qquad j=1, \cdots, \pi,$$

where c is a constant depending on σ and α_{ij} is a $q_i^{n_i}$th root of unity. If $\mathfrak{G}'$ is a subgroup of $\mathfrak{G}$ generated by $\sigma_1^{s_1}, \cdots, \sigma_g^{s_g}$ $(s_i=q_i^{m_i}, 0\leqq m_i<n_i)$ the set $U_{m_1m_2\cdots m_g}$ of integrals u_j which are merely altered by additive constants by the transformations of $\mathfrak{G}'$, and have this property for no larger subgroup of $\mathfrak{G}$, form a complete set of

$$(p-1)\prod_{i=1}^{g}(q_i^{m_i}-q_i^{m_i-1})$$

integrals, provided that in this expression we replace $q_i^{m_i-1}$ by 0 if m_i is zero, and add one if every m_i is zero. It is shown by an example that these systems may be still further reducible. Furthermore, for any assigned set of $q_i^{n_i}$th roots of unity ($i=1, \cdots, g$), not all equal to one, there are $p-1$ integrals for which these are the α_{ij} of (1) (the exceptional case is dealt with above). Special attention is paid to the case of a hyperelliptic Γ.

D. B. Scott.

Scott, D. B. Point-curve correspondences. I. The theory of the base. Proc. Cambridge Philos. Soc. **41**, 135–145 (1945). [MF 12845]

The author considers properties of irreducible algebraic correspondences between irreducible varieties U, V, of respective dimensions r and s, such that to a generic point of U corresponds an algebraic variety of dimension $s-1$ on V and to a generic point of V corresponds an algebraic variety of dimension $r-1$ on U. For such correspondences a number of theorems of Severi and Albanese relating to point-curve correspondences can be easily obtained. The author proceeds to investigate the base for point-curve correspondences between two algebraic surfaces F and G. Using transcendental and topological arguments, it is proved that the number of independent point-curve correspondences is $\rho+\sigma+\lambda$, where ρ and σ are the base-numbers for the algebraic curves on F and G and λ is the simultaneity index of the period matrices of the simple integrals of the first kind.

J. A. Todd (Cambridge, England).

Todd, J. A. Covariant line complexes of a pair of quadric surfaces. Proc. Cambridge Philos. Soc. **41**, 127–135 (1945). [MF 12844]

This paper gives a systematic account of the geometrical origins and relations of the eight quadratic and eight cubic covariant line-complexes of a pair of quaternary quadrics, listed by Turnbull [Proc. London Math. Soc. (2) **18**, 69–94 (1919)]. By a convenient use of matrix notation, the author achieves concise equations for the eight quadratic complexes; he obtains their complete geometric characterization in terms of the two quadrics themselves, two simple covariant quadrics and two corresponding contravariant quadric envelopes; and he obtains explicitly the three quadratic syzygies connecting them. The eight cubic complexes are adequately dealt with at the end.

J. G. Semple.

Châtelet, François. Équivalence de certaines variétés unicursales. C. R. Acad. Sci. Paris **216**, 189–191 (1943). [MF 10951]

This paper describes in general terms a process for studying Poincaré equivalences between certain unicursal vari-

eties. Several results are stated but no proofs are given. [Cf. the same C. R. 216, 142–144 (1943); these Rev. 6, 17.]
W. V. D. Hodge (Cambridge, England).

Godeaux, Lucien. Observations sur les variétés algébriques à trois dimensions sur lesquelles l'opération d'adjonction est périodique. Bull. Soc. Roy. Sci. Liége 9, 2–11 (1940). [MF 13035]

Godeaux, Lucien. Sur les points de diramation des surfaces multiples. Bull. Soc. Roy. Sci. Liége 9, 54–66, 67–79, 128–137 (1940). [MF 13037]

Godeaux, Lucien. Sur un point de la théorie des correspondances entre deux courbes ou deux surfaces algébriques. Bull. Soc. Roy. Sci. Liége 9, 146–151 (1940). [MF 13042]

Godeaux, Lucien. Sur les involutions cycliques régulières appartenant à une surface irrégulière. Bull. Sci. Math. (2) 67, 145–158 (1943). [MF 12636]

If a surface F of irregularity $q>1$ contains a cyclic involution of prime order $p>2$, which is represented by a regular surface Φ with only a finite set U of branch points, then it is shown that there is a surface F_0 (whose irregularity is not less than that of F) which is a normal p-fold multiple of F. The author has previously given a slightly stronger result for $p=2$ [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 13, 394–414, 524–543, 707–724 (1927)]. Two surfaces Ψ_1 and Ψ_2 are found containing involutions of order p represented by Φ and with branch sets U_1 and U_2 respectively, such that $U_i \subset U$ and $U_1 \cup U_2 = U$; F_0 is the image of the (p, p) correspondence set up by Φ between Ψ_1 and Ψ_2.
D. B. Scott (London).

Godeaux, Lucien. Remarque sur l'étude des points unis des involutions cycliques appartenant à une surface algébrique. Bull. Soc. Roy. Sci. Liége 10, 290–295 (1941). [MF 13066]

Godeaux, Lucien. Sur les surfaces du quatrième ordre possédant quatre points doubles uniplanaires. Bull. Soc. Roy. Sci. Liége 9, 151–162 (1940). [MF 13043]

Godeaux, Lucien. Sur les surfaces de bigenre un appartenant à un espace à huit dimensions. Bull. Soc. Roy. Sci. Liége 10, 454–459 (1941). [MF 13073]

Godeaux, Lucien. Sur les surfaces bicanoniques régulières de genre trois. Bull. Soc. Roy. Sci. Liége 10, 498–501 (1941). [MF 13077]

Godeaux, Lucien. Sur les surfaces du quatrième ordre touchant quatre plans le long de quatre droites. Bull. Soc. Roy. Sci. Liége 10, 522–524 (1941). [MF 13078]

Godeaux, Lucien. Sur les surfaces normales de genres un appartenant à un espace linéaire à neuf dimensions. Bull. Soc. Roy. Sci. Liége 10, 525–531 (1941). [MF 13079]

Godeaux, Lucien. Sur une surface bicanonique de genre géométrique égal à un. Bull. Soc. Roy. Sci. Liége 10, 548–553 (1941). [MF 13080]

Burniat, Pol. Sur quelques surfaces irrégulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 666–685 (1941). [MF 12678]

The author determines the arithmetic, geometric and linear genera of certain cyclic irregular surfaces which contain either (i) two algebraic pencils of curves or (ii) an algebraic system of curves of index two. For certain special types of surface the author obtains two theorems of which the following is typical. If Φ is the surface which represents the unordered pairs of points of a curve C^p, it contains an algebraic system $\{C\}$ of curves birationally equivalent to C^p, of index 2 and degree 1. Any surface Φ, containing a cyclic involution J_n^1 of prime order n whose sets are in one-to-one correspondence with the points of Φ, and whose branch curve consists of δ curves of $\{C\}$, is the image of a cyclic involution J_n^2 of order n with δ^2 united points on the surface representing the unordered pairs of points of a cyclic curve nC^p with δ branch points.
J. A. Todd.

Burniat, Pol. Sur des surfaces canoniques. Bull. Soc. Roy. Sci. Liége 9, 192–196 (1940). [MF 13045]

Burniat, Pol. Sur des courbes paracanoniques de S_2. Bull. Soc. Roy. Sci. Liége 10, 120–122 (1941). [MF 13054]

Burniat, Pol. Courbes semi-bicanoniques normales. Bull. Soc. Roy. Sci. Liége 10, 374–378 (1941). [MF 13070]

Frère, André. Sur la hessienne d'une surface algébrique. Bull. Soc. Roy. Sci. Liége 10, 465–468 (1941). [MF 13076]

Derwidué, L. Sur les transformations birationnelles laissant fixes les courbes rationnelles d'une congruence linéaire. Bull. Soc. Roy. Sci. Liége 9, 28–33 (1940). [MF 13036]

Derwidué, L. Sur les transformations birationnelles laissant fixes les courbes elliptiques d'une congruence linéaire. Bull. Soc. Roy. Sci. Liége 9, 162–165 (1940). [MF 13044]

Ledoux, Henri. Sur une transformation birationnelle involutive de l'espace. Bull. Soc. Roy. Sci. Liége 10, 647–653 (1941). [MF 13090]

Maréchal, R. Sur les transformations birationnelles du troisième ordre de l'espace. Bull. Soc. Roy. Sci. Liége 10, 123–135, 192–199 (1941). [MF 13055]

Maréchal, R. Sur une transformation birationnelle (3, 4) de l'espace. Bull. Soc. Roy. Sci. Liége 10, 260–264 (1941). [MF 13065]

Pissard, Nelly. Sur une variété algébrique à trois dimensions. Bull. Soc. Roy. Sci. Liége 10, 643–647 (1941). [MF 13089]

Differential Geometry

Ludwig, Konrad. Die der transversalen Mercatorkarte der Kugel entsprechende Abbildung des Rotationsellipsoids. J. Reine Angew. Math. 185, 193–230 (1943). [MF 12103]

The author studies those conformal maps of the ellipsoid of revolution onto the plane for which one of the principal ellipses is a scale curve; these are generalizations of the transversal Mercator's map of the sphere. Using elliptic functions, he obtains parametric equations for such maps, which express isoperimetric parameters in the ellipsoid and in the plane in terms of a pair of auxiliary parameters.
S. B. Myers (Ann Arbor, Mich.).

Rinner, Karl. Allgemeine Koeffizientenbedingungen in Reihen für konforme Abbildungen des Ellipsoides in der Ebene. Z. Vermessungswesen 73, 102–107 (1944). [MF 12623]

It is stated that, since there exist few practicable formulas for conformal representation of ellipsoids on the plane, it is desirable, especially for military purposes, to have readily applicable rules for computing the coefficients in power series giving these maps. Let φ, q denote geographic and isometric latitudes respectively, let l denote longitude, and let x, y denote the Cartesian coordinates of the image-point in the plane under a mapping which is conformal with respect to (q, l). It is pointed out that, if $D:(q, l)$ is symmetric with respect to a meridian $l=l_0$, and if in the mapping $x+iy=f(q+il)$ the meridian $l=l_0$ corresponds to the x-axis $(y=0)$, then the coefficients in the expansion of f are real, so that the expansion of x in terms of Δq and Δl contains only even powers of Δl, while the expansion of y contains only odd powers of Δl; the series for x contains exactly $m+1$ terms of order $2m$ and the same number of terms of order $2m+1$, while the series for y contains $m+1$ terms of each of these orders; all $n+1$ terms of order n are determined when one of them is known. Substitution of the series expansion of Δg in terms of $\Delta\varphi$ gives rules for the expansion of x, y in terms of $\Delta\varphi$, Δl analogous to rules for their expansion in terms of Δq, Δl. *E. F. Beckenbach.*

Strubecker, Karl. Über die Flächentreuen Abbildungen der Ebene. Bull. Math. Soc. Roumaine Sci. 44, 59–70 (1942). [MF 12750]

The basic result is that a correspondence exists between area-preserving transformations of the plane and certain mappings (parataktische Abbildungen) of a surface element in isotropic space. A mapping of this sort is analogous to the spherical representation of a surface in ordinary space. The result was first stated by Scheffers [Math. Z. 2, 180–186 (1918)] without its present geometric context. The author also develops various analogues of classical differential geometry which apply to the geometry of a surface in isotropic space. *A. Fialkow* (New York, N. Y.).

Bol, G. Über Nabelpunkte auf einer Eifläche. Math. Z. 49, 389–410 (1944). [MF 11996]

Carathéodory's conjecture, that on every (analytic) closed surface of genus 0 there exist at least two umbilical points, was first proved by H. Hamburger [Ann. of Math. (2) 41, 63–86 (1940); Acta Math. 73, 175–228, 229–332 (1941); these Rev. 1, 172; 3, 310]. The present paper contains a new proof. The theorem is again proved by considering the singularity in the net of lines of curvature at the umbilic, and takes the following form. With an analytic function $w(\rho, \theta)=\sum_{m=k}^{\infty}\rho^m w_m(\theta)$ $(k\geqq 3)$, with period 2π in θ, form, in an (x, y) plane, the curves C_ρ:

$$x=-\rho^2 w_{\rho\rho}+\rho w_\rho+w_{\theta\theta}, \quad y=2(\rho w_{\rho\theta}-w_\theta),$$

ρ arbitrary but fixed, $0\leqq\theta\leqq 2\pi$; assume that no C_ρ goes through $(0, 0)$; then each C_ρ has a nonnegative index with respect to $(0, 0)$.

For the proof, approximations to C_ρ are used. The first approximation consists in taking only the terms with lowest ρ-exponents in x and y. If this curve does not go through $(0, 0)$ the index is shown to be nonnegative because of the particular form of the equations obtained. In the contrary case, there exist Puiseux series solutions of $x=0$ and $y=0$ for θ in terms of ρ, beginning with the same constant term θ_0. Generalizing this, a "singularity of degree k" means that there exist such Puiseux series solutions which coincide in the first $k+1$ terms; call the sum of these terms $\varphi(\rho)$. Then a new parameter s is introduced by $\theta=\varphi(\rho)+s\cdot\rho^s$, where s is the lowest exponent of the term of order $k+2$ in any of the Puiseux series mentioned. In terms of s one obtains a new approximation for a segment of the original curve. If this does not go through $(0, 0)$, the index is again shown to be nonnegative because of the form of the equations. If it does go through $(0, 0)$, a singularity of degree $k+1$ results. Singularities of arbitrarily high degree cannot exist, since then $x=0$ and $y=0$ would have a common Puiseux series solution for θ in terms of ρ, and every C_ρ would go through $(0, 0)$, contrary to assumption. [On page 402 a factor s has been lost in equations (61), (62) and (63).] *H. Samelson* (Syracuse, N. Y.).

Busemann, H., and Feller, W. Regularity properties of a certain class of surfaces. Bull. Amer. Math. Soc. 51, 583–598 (1945). [MF 12820]

The total curvature of a plane curve is defined as $\int_0^{2\pi}N(\varphi)d\varphi$, where $N(\varphi)$ is the number of points at which there exists a tangent making an angle φ with an assigned direction. (By tangent at a point of parameter t_0 is meant any directed line obtainable as a limit of directed chords $t_n\tau_n$, where $t_n\uparrow t_0$ and $\tau_n\downarrow t_0$.) It is shown that a curve of finite total curvature is rectifiable and has, almost everywhere with respect to length, a unique tangent and defined curvature. Hence it is shown that, if a surface $z=f(x, y)$, where f satisfies a Lipschitz condition, is such that the sections $x\sin\alpha+y\cos\alpha=p$ have uniformly bounded total curvature for $\alpha=0$ and α_1 or $-\alpha_1$, with p arbitrary in some range, then, almost everywhere, f has a second differential and the sections of the surface have defined curvatures satisfying the relations of Euler and Meusnier. *A. J. Ward.*

Germain, Paul. Sur divers points de géométrie infinitésimale et sur l'application des formules de Lelieuvre à l'étude d'une famille de surfaces. Revue Sci. (Rev. Rose Illus.) 79, 69–84 (1941). [MF 12859]

Let $\mathfrak{x}=\mathfrak{x}(\alpha, \beta)$ be the parametric representation of a surface of negative Gaussian curvature K, the asymptotic lines being the parameter curves. The "Lelieuvre vector" $\omega=\omega(\alpha, \beta)$ is a vector of the length $(-K)^{-\frac{1}{4}}$ normal to the surface. By means of this vector, Lelieuvre's formulas can be written in the form $\mathfrak{x}_\alpha=\omega\times\omega_\alpha$, $\mathfrak{x}_\beta=-\omega\times\omega_\beta$. Thus ω satisfies the condition of integrability $\omega\times\omega_{\alpha\beta}=0$. These formulas enable the author to prove various classical theorems in a simple way and to add some new results. Thus the Λ-surfaces are studied and characterized; these are surfaces satisfying the equation $\omega_{\alpha\beta}=0$ or $\omega=\mathfrak{y}(\alpha)+\mathfrak{z}(\beta)$. The special case $\mathfrak{y}^2=\mathfrak{z}^2=1$ defines the L-surfaces. They are identical with the isometric images of imaginary rotational paraboloids $x^2+y^2=2ipz$ $(p>0)$. The surface $\omega=\omega(\alpha, \beta)$ is called the "indicatrix of the Gaussian curvature" of the surface $\mathfrak{x}(\alpha, \beta)$. The indicatrices of the Λ-surfaces, of the surfaces of constant negative curvature and of the ruled surfaces are discussed. The surface $\omega(\alpha, \beta)$ is a ruled surface if and only if $\mathfrak{x}(\alpha, \beta)$ is one. The validity of the results on Λ-surfaces is analyzed assuming only that $\mathfrak{y}$ and $\mathfrak{z}$ (1) have continuous first derivatives or (2) are continuous and of bounded variation. *P. Scherk* (Saskatoon, Sask.).

Chern, Shiing-shen. Some new characterizations of the Euclidean sphere. Duke Math. J. 12, 279–290 (1945). [MF 12599]

The "indeformability" of the sphere is contained in the theorem that a closed surface of constant Gaussian curva-

ture in three-dimensional Euclidean space is a sphere. It is also known that a closed convex surface of constant mean curvature is a sphere. One of the main results obtained by the author contains both of these theorems as special cases. A surface with continuous third derivatives and no singularities is called a W-surface if its principal curvatures r_1 and r_2 satisfy $\lambda_1 dr_1+\lambda_2 dr_2=0$ with λ_1 and λ_2 not both zero. A W-surface is called special if λ_1 and λ_2 are both positive. The author shows that a closed convex special W-surface is a sphere; the above two theorems are immediate consequences. Further consequences are that a convex surface is a sphere if $K=cH^2$ or if $aK+bH+c=0$. Here K and H are the Gaussian and mean curvatures, respectively, and a, b and c are constants not all of which are zero.

The theorem is proved under the assumption that the special W-surface is convex. It is not known whether the restriction to convex surfaces is necessary, but the author gives two further theorems which indicate that counterexamples would not be very simple. (1) A closed analytic surface is a sphere if $ar_1+br_2+c=0$, with a, b nonzero constants. In particular, a closed analytic surface of constant mean curvature is a sphere. (2) A closed analytic surface of revolution of genus zero is a sphere if it is a special W-surface.

These results are in the field of differential geometry in the large. The author obtains also a number of results valid in the small. A surface is said to be a surface of Bonnet if it can be mapped isometrically on another surface with preservation of the principal curvatures in such a way that the mapping is neither a rigid body motion nor a reflection. The author shows that the only convex surface which permits an isometric mapping with preservation of the principal curvatures is the sphere, with the consequence that a convex surface cannot be a surface of Bonnet. [Reviewer's remark. The statement of this result in the author's theorem 2, p. 280, requires a slight correction and amplification.] Two earlier results of H. Weyl are obtained from the author's formulas. (1) If $e(K)$ is the second differential parameter of the Gaussian curvature K, then the maximum of $K-e(K)/4K$ is an upper bound for the mean curvature of a convex surface. (2) On a convex surface a relative maximum of the mean curvature and a relative minimum of the Gaussian curvature can occur together only at umbilical points. Most of the above results are obtained with surprising ease from the differential equations of surface theory, particularly the Codazzi equations, expressed in the symbolism introduced by É. Cartan. *J. J. Stoker.*

Choquet, Gustave. Caractérisation de la sphère en géométrie infinitésimale directe. Revue Sci. (Rev. Rose Illus.) **81**, 447–452 (1943). [MF 12716]

A well-known theorem states that, if all the points of a surface S are umbilics, then S is part of a sphere or of a plane. The proof usually assumes that the functions defining S have continuous second derivatives. In this paper these assumptions are analyzed from the point of view of Bouligand's "géométrie infinitésimale directe." If P is a point of the surface S then the paratingent Π_P of S at P is the set of the limits of all the straight lines $P'P''$ as the two different points P', P'' converge to P on S. Let all the Π_P's be planes. Thus, if a part of S is given by an equation $z=f(x, y)$, the first derivatives of $f(x, y)$ exist and are continuous. The point P of S is called an umbilic if the sphere through a second point $P'\neq P$ of S and tangent to Π_P at P has a unique limit as P' converges to P. With these assumptions and definitions the author proves the above theorem. The main difficulty of his proof lies in excluding points P where the radius of the limit sphere is zero or infinity. He overcomes it by observing that this radius is a function of Baire of the first class. The special case where S is a convex surface and the radius of the limit sphere is assumed different from zero everywhere was discussed by Busemann and Feller [Acta Math. **66**, 1–47 (1936)].

In the second part of his paper, the author improves the above result. Let S be a point set in 3-space, homeomorphic to a plane set consisting of a domain and its boundary. If P is a point of S, the contingent of S at P consists of all the limits of the rays PP' as the point $P'\neq P$ converges to P on S. We assume that the contingent of S at each point P lies in a plane Π_P that varies continuously with P. Replacing Π_P by Π_P' we can again define umbilics. Let every point of S whose contingent is a whole plane be an umbilic. Then S is part of a sphere or of a plane. Some additional applications of the theory of Baire's classes to the "géométrie infinitésimale directe" are indicated. *P. Scherk.*

Thomas, T. Y. Algebraic determination of the second fundamental form of a surface by its mean curvature. Bull. Amer. Math. Soc. **51**, 390–399 (1945). [MF 12518]

The principal result of this paper is that in general the components of the second fundamental tensor of a two-dimensional surface imbedded in three-dimensional Euclidean space can be expressed as algebraic functions of the metric tensor of the surface, its mean curvature, and their derivatives. Explicit formulae for these functions are given, valid provided a rather complicated scalar denominator (called W) does not vanish. In two special cases the case $W=0$ is treated completely. (1) If W vanishes over a closed surface of positive Gaussian curvature, the mean curvature and the Gaussian curvature are constant over the surface. (2) On a closed analytic surface which is of positive but not constant Gaussian curvature, W can vanish only at exceptional points.

The inverse problem may be formulated as follows. Given a two-dimensional abstract Riemannian manifold, what are the conditions which must be satisfied by a scalar function H so that it may be considered to be the mean curvature of an imbedding of this manifold in Euclidean 3-space? This problem is solved, the solution being expressed as a pair of inequalities and a pair of differential equations involving H and the metric tensor. If H satisfies these conditions, the imbedding is completely determined. The importance of this result is that the question of imbedding a surface in Euclidean 3-space now rests in general upon the determination of a single scalar H instead of on the determination of the three components of the second fundamental tensor. *C. B. Allendoerfer* (Haverford, Pa.).

Cartan, Élie. Les surfaces qui admettent une seconde forme fondamentale donnée. Bull. Sci. Math. (2) **67**, 8–32 (1943). [MF 12627]

This is a detailed treatment of results announced previously [C. R. Acad. Sci. Paris **212**, 825–828 (1941); these Rev. **3**, 17]. The author studies the problem of surfaces in ordinary space having a given second fundamental form, a problem analogous to that of applicability of surfaces. The problem is formulated analytically by the method of moving frames and it is proved that the general solution depends on three arbitrary functions in one variable. Among the consequences drawn from the study of the differential system is an analogue of Gauss's theorem which expresses the

Riemannian curvature of the second fundamental form in terms of the principal curvatures of the surface and their covariant derivatives along the principal directions. The problem of Cauchy is studied and the singular solutions are discussed. The whole discussion is divided into two cases, according as the second fundamental form is definite or indefinite. *S. Chern* (Princeton, N. J.).

Long, Louis. Recherches de géométrie infinitésimale. Systèmes de formules dans l'espace à cinq dimensions pour la représentation des surfaces rapportées à leurs lignes de courbure. Bull. Soc. Roy. Sci. Liége 10, 381–388 (1941). [MF 13072]

Long, Louis. Recherches de géométrie infinitésimale. Systèmes de formules dans l'espace à huit dimensions pour la représentation des couples de surfaces applicables. Bull. Soc. Roy. Sci. Liége 10, 574–583 (1941). [MF 13084]

Arghiriade, E. Sur les cônes quadratiques osculateurs d'une courbe gauche. Bull. Math. Soc. Roumaine Sci. 42, no. 2, 3–8 (1940). [MF 12742]

In previous papers [Ann. Sci. Univ. Jassy 24, 465–510 (1938); 27, 31 ff. (1942)] the author has shown that the skew curve called the biaxial spiral of the first kind, $x_1=\cos u$, $x_2=\sin u$, $x_3=\cosh u$, $x_4=\sinh u$, possesses at each of its points two 8-point quadric cones, the locus of whose vertices is a straight line. The locus of the vertices of the 7-point quadric cones is a skew curve of order six [E. P. Lane, Projective Differential Geometry of Curves and Surfaces, University of Chicago Press, 1932, p. 22]. In order that a curve have 8-point quadric cones with their vertices on a line, this sextic curve must be tangent to the line. Using these and other ideas, all curves with the desired property are found. *V. G. Grove* (East Lansing, Mich.).

Arghiriade, E. Sur la géométrie biaxiale des courbes gauches. Bull. Math. Soc. Roumaine Sci. 44, 3–19 (1942). [MF 12746]

The study of the differential properties of skew curves invariant under the subgroup of the projective group which preserves two nonintersecting lines has been called the biaxial geometry of skew curves [O. Mayer, Bull. Math. Soc. Roumaine Sci. 40, 193–196 (1938)]. If the lines are taken as $x_1=x_2=0$, $x_3=x_4=0$, all biaxial transforms of a given curve are defined by the differential equations

$$x_i''=p_1x_i'+r_1x_i, \qquad i=1, 2,$$
$$x_j''=p_2x_j'+r_2x_j, \qquad j=3, 4.$$

Curves for which $p_1=p_2$ are called biaxial isotropic curves. A study is made of these curves, several special cases being found. The biaxial spirals of the first kind are the only curves possessing a family of 8-point cones whose vertices lie on a straight line [see the preceding review]. There exists a curve, called a biaxial spiral of the second kind, which belongs to a linear complex and has a family of 7-point cones with their vertices on a straight line. *V. G. Grove* (East Lansing, Mich.).

Arghiriade, E. Sur la ligne flecnodale d'une surface non reglée. Bull. Math. Soc. Roumaine Sci. 45, 55–62 (1943). [MF 12756]

An asymptotic tangent line of a surface $S: z=f(x, y)$ in ordinary space intersects S at the point p at which it is constructed in three consecutive points. The point p is termed flecnodal if one of the asymptotic tangents at p intersects S in four consecutive points. On a ruled surface, every point is flecnodal, whereas on a nonruled surface, there is a single flecnodal locus. The author proves the following theorem of Halphen. If, at a point p of S, there exists a quadric Σ having third order contact with S, the straight lines of Σ which pass through p have third order contact with S. The point p is a double point of the flecnodal locus of S. It is shown that the converse of this theorem is not valid in general (as thought by Halphen). If p is a parabolic point or an elliptic point in the real domain, and if p is a double point of the flecnodal locus, then there exists a quadric Σ having third order contact with S at p. For a hyperbolic point, p can be a double point of the flecnodal locus and there is, in general, no hyperosculating quadric Σ to S at p. The points where the surface has contact of third order with a quadric cone are the double points of the locus of parabolic points, and conversely. A double point of the parabolic locus is a double point of the flecnodal locus, and conversely. *J. DeCicco* (Chicago, Ill.).

Charrueau, André. Sur la courbure et la torsion géodésique. Bull. Sci. Math. (2) 67, 33–41 (1943). [MF 12628]

The author reestablishes a theorem originally due to Scheffers [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 50, 261–294 (1898)], concerning isogonal trajectories on a surface. This theorem states that the centers of geodesic curvature of those curves of an isogonal family on a surface which pass through the same point lie on a straight line. Various applications of this theorem to geometry and mechanics are given. *A. Fialkow* (New York, N. Y.).

Vincensini, Paul. Sur une transformation de l'espace réglé et sur les systèmes sphériques isothermes. Bull. Sci. Math. (2) 65, 155–178, 186–210 (1941). [MF 12836]

The transformation in question is the following. Let D be an arbitrary line in Euclidean 3-space with direction cosines X, Y, Z; let H be the orthogonal projection of the origin O on D; let K be the orthogonal projection of H on the xy plane. Now turn the vector OK about O in the xy plane through an angle of $+\pi/2$ and subject it to a homothetic transformation with center O and ratio $F(Z)$, where $F(Z)$ is an arbitrary function of Z. Let J' be the extremity of the resulting vector. Draw through J' the line D' parallel to D. Then D' is defined as the transform of D by the transformation $T[F(Z)]$. By means of this transformation the author has succeeded in synthesizing a number of known results on congruences and has developed additional properties.

Special attention is given to normal congruences, the properties of their transforms and the conditions under which the normality of a congruence is preserved by the transformation. Among the transforms of normal congruences are (1) all congruences whose mean planes pass through the origin (generalized congruences of Appell), (2) all congruences having a plane mean surface and (3) all congruences whose associated foci are equidistant from a fixed line. The transformation $T[F(Z)]$ may be decomposed into the product of a rotation of D through $+\pi/2$ about its parallel through O and a homothetic transformation of center O and ratio $\varphi(Z)$. This leads to a study of subgroups of the full group of $T[F(Z)]$. Applications are made to the theories of orthogonal isotherms on the sphere and of minimal surfaces. These depend upon the properties of the surfaces of Appell, which are the normal surfaces to the congruences of Appell described above. *C. B. Allendoerfer* (Haverford, Pa.).

Mihăileanu, N. N. Sur les tissus plans de première espèce. Bull. Math. Soc. Roumaine Sci. 43, 23–26 (1941). [MF 12726]

Pantazi [C. R. Acad. Sci. Roum. 2, 108–111 (1938)] has studied a class of n-webs of curves in the plane, which are called n-webs of the first kind and are characterized by the vanishing of a differential invariant. This invariant is calculated explicitly. *S. Chern* (Princeton, N. J.).

Marcus, Eph. Sur les cycles de Laplace de période quatre. Mathematica, Timişoara 20, 23–28 (1944). [MF 12450]

A configuration (T) is the set of four congruences generated by the sides of a skew quadrilateral the loci of whose vertices are the focal surfaces of the congruences. Among these configurations are the sequences of period four of the transformations of Laplace. The associated quadrilateral is called a quadrilateral of Laplace. Two congruences are said to be stratifiable if there exists a single infinitude of transversal surfaces of one congruence such that the tangent planes to each of the surfaces at its point of intersection with the lines of that congruence pass through the corresponding lines of the second congruence. This paper studies quadrilaterals of Laplace, using the classical method and notations of Darboux. Representative theorems may be stated as follows. If the diagonals of a quadrilateral of Laplace are stratifiable, then the opposite vertices generate autoprojective nets; conversely if, in a sequence of Laplace of period four, two opposite vertices generate autoprojective conjugate nets, then the diagonals generate stratifiable congruences. If a congruence is such that the asymptotic net on one focal surface corresponds to a conjugate net on the other, the congruence is called a Waelsch congruence. Then if, in a quadrilateral of Laplace, one side generates a Waelsch congruence, the opposite does likewise; and if two consecutive edges generate Waelsch congruences, the other two edges do likewise. *V. G. Grove* (East Lansing, Mich.).

Sprague, Atherton H. Surfaces whose lines of curvature are nets R, and their transformations. Revista Ci., Lima 47, 3–32 (1945). [MF 12706]

This paper discusses, by classical methods, surfaces the tangents to whose lines of curvature generate W congruences. Among such surfaces are those with constant Gaussian curvature. *V. G. Grove* (East Lansing, Mich.).

Rozet, O. Recherches sur les congruences de droites. Bull. Soc. Roy. Sci. Liége 10, 138–167 (1941). [MF 13056]

This paper is a coordination and occasionally an extension of results in the analytic study of linear congruences in three dimensions due in the main to Wilczynski, Mentré, Terracini, and Finikoff. First the general formulas are so presented as to facilitate derivation of known results and discovery of extensions. They are applied to the study of the Laplace series defined by the focal net. A treatment of congruences through the Klein five dimensional representation follows. It is used to establish properties of certain complexes which Waelsch associated with a congruence. A chapter on projective deformation of congruences completes the study. *J. L. Vanderslice* (College Park, Md.).

Rozet, O. Sur les complexes de droites dont un foyer inflexionnel décrit une surface non réglée. Bull. Soc. Roy. Sci. Liége 10, 296–300 (1941). [MF 13067]

Rozet, O. Sur les complexes de droites formés par les génératrices rectilignes des quadriques de Lie d'une surface. Bull. Soc. Roy. Sci. Liége 10, 379–381 (1941). [MF 13071]

Rozet, O. Sur les complexes d'accompagnement de Waelsch. Bull. Soc. Roy. Sci. Liége 10, 559–561 (1941). [MF 13082]

Rozet, O. Sur certains systèmes-points de Guichard et sur certains complexes de droites. Bull. Soc. Roy. Sci. Liége 10, 612–622 (1941). [MF 13086]

Rozet, O. Sur un complexe de droites du troisième ordre. Bull. Soc. Roy. Sci. Liége 10, 622–625 (1941). [MF 13087]

Tikhotzky, C. La transformation K des complexes. Rec. Math. [Mat. Sbornik] N.S. 16(58), 87–100 (1945). (French. Russian summary) [MF 13016]

This is a generalization to line complexes of the author's previous work [same Rec. N.S. 5(47), 297–305 (1939); these Rev. 2, 18] on the deformation of line congruences in a Euclidean space of three dimensions. Two line complexes C and C' are called K-transformable if their lines can be put into a one to one correspondence such that to each pair of corresponding lines L and L' there exists a motion which carries L' into L and every ruled surface of C' containing L' into a ruled surface tangent to the corresponding ruled surface of C along L. This notion of deformation was first introduced by É. Cartan. Using Cartan's method of moving frames, the author proves that the line complexes which are K-transformable into a given one depend, in general, on four arbitrary functions of one variable. *S. Chern.*

Finikoff, S. Coupie de surfaces linéaires stratifiables par deux familles de courbes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 79–112 (1945). (Russian. French summary) [MF 12767]

A pair of ruled surfaces L and L' whose generators are in a one to one correspondence is said to be stratifiable if one parameter families C and C', respectively, of curves on L and L' can be given so that the osculating planes to all curves of C at their points of intersection with a generator of L go through the corresponding generator of L', and vice versa. The paper is devoted to the investigation of relations between such pairs of surfaces and other configurations of projective differential geometry such as systems of Bianchi, congruences W, stratifiable pairs of curves and stratifiable pairs of congruences. The main result is that a pair of stratifiable surfaces may always be imbedded in a stratifiable pair of congruences [cf. Rec. Math. [Mat. Sbornik] N.S. 12(54), 287–314 (1943); these Rev. 6, 19]; when one of the surfaces is developable the other is also, and in this case the imbedding pair of congruences is determined uniquely. Whereas earlier papers dealing with stratification involved metric considerations, here the discussion is entirely projective. The method is that of Grassmann and Cartan; however, since the main imbedding theorem deals with a case which, because of the appearance of singular elements, is not covered by the existence theorem of Cartan, recourse is made to the general theory of partial differential equations as given by Riquier. *G. Y. Rainich.*

Pantazi, Al. Sur la déformation projective des quadruples stratifiables. Bull. Math. Soc. Roumaine Sci. 42, no. 2, 49–67 (1940). [MF 12744]

The author showed in a previous paper [Ann. Roumaines Math. 2, 3–34 (1935)] that the conjugate quadruples are of three types and studied some of their properties. With the help of methods developed by Finikoff [J. Math. Pures Appl. (9) 18, 405–415 (1939); these Rev. 1, 173] the author now proves, in particular, the following results. (1) Class C breaks up into two subclasses. The first consists of ∞^4 families of ∞^3 quadruples; the second consists of ∞^5 families of ∞^1 quadruples. The members of each family are projectively applicable to one another. (2) The quadruples of class B are all deformable and applicable to a subclass of quadruples common to classes A and B. (3) The quadruples admitting projective transformations into themselves are members of two special subclasses of class C.

A. Schwartz (State College, Pa.).

Pantazi, Al. Correspondance entre deux surfaces à axes confondus. Bull. Math. Soc. Roumaine Sci. 42, no. 1, 57–63 (1940). [MF 12738]

Čech has shown that, when a point correspondence in projective three-space is established between two surfaces S and S_1, there exists for each pair of corresponding points A and A_1 a pair of lines a and a_1 (and in general only two), passing, respectively, through A and A_1, such that to every curve which passes through A and whose osculating plane at A contains a there corresponds a curve whose osculating plane at A_1 passes through a_1. The present paper investigates the special case in which the lines a and a_1 coincide with AA_1. It is shown that the determination of S and S_1 having this property depends on five arbitrary functions of a single argument. Equations are developed for the focal surfaces of the congruence generated by AA_1. Furthermore, this congruence is identical with that generated by the intersection of the osculating planes of two skew curves C and C_1. Inversely, one may choose any two skew curves C and C_1 and then the determination of the corresponding surfaces S and S_1 depends on one arbitrary function of a single argument. Differential equations are given for S and S_1; the integration of these is deferred to a later paper.

C. B. Allendoerfer (Haverford, Pa.).

Pantazi, Al. Sur la déformation projective des surfaces non holonomes de l'espace E_3. Bull. Math. Soc. Roumaine Sci. 45, 33–47 (1943). [MF 12754]

A definition of the projective deformation of nonholonomic surfaces of three-dimensional projective space is suggested, by considering the osculating planes and the limit points (the duals of osculating planes) of curves of the surface. Two such surfaces are called projectively deformable if a one-to-one correspondence can be established between their points such that, for every pair of corresponding points, there exists a projective transformation which carries the osculating planes and the limit points of the curves of one surface into those of the corresponding curves of the other. The differential system for the projective deformability of two nonholonomic surfaces is established and differential invariants are deduced which remain unaltered under projective deformation. In particular, it is shown that the asymptotic curves are preserved. However, it is not known whether there exists a nonholonomic surface which can be projectively deformed without remaining projectively equivalent to itself. [Another kind of projective deformation of nonholonomic surfaces, based on a generalization of the projective arc, was studied by Wang [Ann. of Math. (2) 44, 562–571 (1943); these Rev. 5, 15].]

S. Chern.

Mihăilescu, Tiberiu. Sur les variétés non holonomes de l'S_3 projectif. Bull. Math. Soc. Roumaine Sci. 44, 35–53 (1942). [MF 12748]

The differential geometry of nonholonomic surfaces in three-dimensional projective space is developed by the method of moving frames of É. Cartan. Under the assumption that the asymptotic directions are distinct, the frames are specified to the extent that there is a three-parameter family of frames at each point. Equations for the infinitesimal displacement of the frames and integrability conditions follow from the corresponding equations of the projective group and are given explicitly. As generalizations of classical concepts for holonomic surfaces, definitions are given of the Darboux and Segre directions, the directrices of the first kind, etc.

S. Chern (Princeton, N. J.).

Mihăilescu, Tiberiu. Sur les variétés non holonomes paraboliques. Bull. Math. Soc. Roumaine Sci. 45, 139–155 (1943). [MF 12764]

Continuing the paper reviewed above, the author studies the nonholonomic surfaces whose asymptotic directions coincide at each point. Such surfaces are classified into three types characterized, respectively, by the conditions: (1) the asymptotic curves are not straight lines; (2) the asymptotic curves are straight lines and the cone of Bompiani consists of a pair of distinct planes; (3) the asymptotic curves are straight lines and the cone of Bompiani is a double plane. The degrees of generality of such surfaces are given. In showing the existence of surfaces of type (1) the author corrects an erroneous statement of Voss [Math. Ann. 23, 45–82 (1884)] that such surfaces do not exist.

S. Chern (Princeton, N. J.).

Wagner, V. Differential geometry of the family of R_k's in R_n and of the family of totally geodesic S_{k-1}'s in S_{n-1} of positive curvature. Rec. Math. [Mat. Sbornik] N.S. 10(52), 165–212 (1942). (English. Russian summary) [MF 12831]

The theory of rectilinear congruences in Euclidean 3-space R_3 is known to be related to the geometry of a nonholonomic V_3^2 in R_3. The author investigates a broad generalization of this connection based on m-parameter families $R_n^{k,m}$ of flat k-spaces R_k in R_n. Such a family determines at each point of a V_{k+m} (the locus of the R_k's) an m-dimensional pencil of directions, thus defining a V_{k+m}^m in V_{k+m} which in general is nonholonomic. Also, $R_n^{k,m}$ has at each point of R_n a conical image formed by translating all R_k's to pass through the point. The geometry of this conical image is equivalent to that of an m-parameter family of totally geodesic S_{k-1}'s in S_{n-1} of constant positive curvature. The latter is in turn connected with the geometry of a nonholonomic V_{k-1+m}^m in V_{k-1+m}. The differential geometry of these several related disciplines is developed in considerable detail using the Schouten idiom. There follow various specializations: congruences of R_k's in R_n, rectilinear congruences in R_n and a consequent generalization of the Kummer theory, geodesic congruences in S_3 and the allied rectilinear congruences in elliptic 3-space.

J. L. Vanderslice.

Thomas, T. Y. On the projective theory of two dimensional Riemann spaces. Proc. Nat. Acad. Sci. U. S. A. 31, 259–261 (1945). [MF 13293]

A projective Riemann space is the totality of Riemann metrics which yield the same system of geodesics. The

previous theory of such spaces includes only one significant projective invariant, namely, the Weyl projective curvature tensor. In this paper a new invariant for two-dimensional projective Riemann spaces is introduced, namely, the relative vector $gg^{\alpha\beta}K_\beta$ of weight two, where K_β is the first covariant derivative of the Gaussian curvature. This leads to a new proof of Beltrami's theorem that the only spaces whose geodesics are identical with the geodesics of a space of constant curvature are spaces of constant curvature. Other properties of this invariant remain to be developed.
C. B. Allendoerfer (Haverford, Pa.).

Botella Raduán, F. On the foundations of the intrinsic geometry of a Riemann space and the properties of the moving reference system. Revista Mat. Hisp.-Amer. (4) 1, 163–170 (1941). (Spanish) [MF 12996]

In a space of affine connection let a set of fundamental vectors e_i (in the sense of Cartan) be defined as well as an absolute differential $De_i = de_i + \Gamma_{ij}{}^k e_k dx^j$, whose $\Gamma_{ij}{}^k$ are not assumed to be symmetric in i, j. The author states that the space is a Riemann space (symmetric $\Gamma_{ij}{}^k$) if (1) $e_j \cdot De_i + e_i \cdot De_j = 0$ and (2) $\partial e_i / \partial x^j = \partial e_j / \partial x^i$. The proof fails because of an error in sign in equation (7). Under assumptions (1) and (2) the theorem is, in fact, false. The error is equivalent to replacing (1) by (3) $e_j \cdot De_i = 0$. Assumption (3) is more natural than assumption (1), and on the basis of (2) and (3) the theorem becomes true.
C. B. Allendoerfer (Haverford, Pa.).

Botella Raduán, F. Note on the foundations of the intrinsic geometry of a Riemannian space. Comment on a review. Revista Mat. Hisp.-Amer. (4) 4, 10–15 (1944). (Spanish) [MF 12189]

The paper reviewed above was found to be in error by Maxia [Zentralblatt für Math. 26] and by the present reviewer. This paper is a rebuttal to Maxia's review and a reaffirmation of the truth of the theorem. The proof is still incorrect; the error occurs on pages 13 and 14, the exact point depending on which statements are interpreted as misprints and which as errors. *C. B. Allendoerfer.*

Moisil, Gr. C. Sur les géodésiques des espaces de Riemann singuliers. Bull. Math. Soc. Roumaine Sci. 42, no. 1, 33–52 (1940). [MF 12736]

An investigation is made of the properties of singular Riemann spaces, that is, those whose metric tensor g_{ij} is of rank $r < n$. Previous papers on this subject have discussed the "integrable" case in which there exists a coordinate transformation which reduces the original degenerate quadratic form in n variables to a nondegenerate quadratic form in r variables and $n-r$ parameters. The author calls such spaces "regular"; most of his paper is concerned with the irregular case, which is the general one. The most important question discussed is that of the geodesics in such spaces. It is shown that these lie on a variety which is the intersection of a number of quadratic varieties. By a method due to Routh, simplified differential equations of the geodesics are obtained. If the least linear space containing the set of quadratic varieties just mentioned is of dimension t, the singular Riemann space is said to be of species $n-t$. In general, only spaces of species zero admit linear connections. Parallelism, however, is defined in all singular Riemann spaces by means of the original metric tensor. The paper concludes with a few remarks on regular spaces. [The bibliography omits one relevant paper: T. Y. Thomas, Proc. Nat. Acad. Sci. U. S. A. 20, 215–219 (1934).]
C. B. Allendoerfer (Haverford, Pa.).

Roşca, Radu M. Transformations de Bäcklund des courbes à torsion constante dans l'espace elliptique. Bull. Math. Soc. Roumaine Sci. 43, 45–57 (1941). [MF 12728]

Bäcklund transformations of curves of constant torsion in ordinary space are defined (by Bianchi) as asymptotic transformations such that the transformed curve has the given constant torsion and the distance between corresponding points is constant. The author discusses such transformations in elliptic space and outlines the three cases into which the theory separates. He has treated some of these questions before [J. Math. Pures Appl. (9) 18, 167–215 (1939); these Rev. 1, 84]. He proves a theorem of permutability for these transformations similar to the standard theorem for general Bäcklund transformations. Configurations of Möbius of types 4_3, 8_4 and higher are displayed.
C. B. Allendoerfer (Haverford, Pa.).

Wagner, V. The generalization of Ricci's and Bianchi's identities for a connexion in the compound manifold. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 303–305 (1945). [MF 12949]

The author considers a compound manifold with a connexion of class s (that is, the coefficients of the connexion involve derivatives up to the sth order inclusive) and studies the effect of the displacement of vectors along infinitesimal parallelograms and parallelepipeds. In this way the identities of Ricci and Bianchi are generalized. *S. Chern.*

Vranceanu, G. Sur l'équivalence en géométrie. Bull. Math. Soc. Roumaine Sci. 42, no. 2, 69–97 (1940). [MF 12745]

In an earlier work, the writer has shown that it is possible to associate an affine connection with any group that leaves two or more complementary Pfaffian systems invariant. This result constitutes a generalization of Cartan's theorem on the equivalence of Pfaffian forms. The present paper applies the author's theorem to the problems of equivalence of affinely connected spaces and of separable groups and to the determination of the invariants of a system of second-order differential equations. *A. Fialkow.*

Vranceanu, G. Sur les espaces à connexion affine et projective. Bull. Math. Soc. Roumaine Sci. 44, 85–139 (1942). [MF 12752]

The author investigates the holonomic and nonholonomic subspaces of a space having an affine connection or a projective connection. The determination of a complete set of invariants for these subspaces is related to various questions considered in the paper reviewed above. The coefficients of the connection are introduced at each point of the space as the coefficients of a (projective or affine) coordinate transformation between local coordinate systems at nearby points. Some of the results are known. *A. Fialkow.*

Vranceanu, G. Sur la théorie des espaces à connexion conforme. Bull. Math. Soc. Roumaine Sci. 45, 3–31 (1943). [MF 12753]

The author studies the geometry of spaces defined by a conformal connection. Among the topics considered are hyperspherical coordinates, similitudes and inversions, conformal groups, associated projective space, torsion and curvature tensors and rigid conformal connections. The method is an extension of the method of normal congruences used in the geometry of Riemann spaces.
A. Fialkow (New York, N. Y.).

Haimovici, A. Sur une généralisation des surfaces développables dans les espaces conformes. Bull. Math. Soc. Roumaine Sci. 45, 125–137 (1943). [MF 12763]

In 1869, Enneper showed that the envelope of the osculating spheres of any curve has properties analogous to those of developable surfaces. More generally, one can consider the envelope of ∞^1 spheres as a generalization of developable surfaces in conformal space. The author gives a definition of conformal developables by use of a property analogous to that of the applicability of ordinary developables upon a plane. Let S and Σ be the two nappes of the envelope of a congruence of spheres (Γ), $\bar{S}$ and $\bar{\Sigma}$ the corresponding ones of the envelope of a second congruence $(\bar{\Gamma})$. Let M and N be the points of contact of any sphere A of (Γ) with S and Σ, and let $\bar{M}$ and $\bar{N}$ be those of a sphere $\bar{A}$ of $(\bar{\Gamma})$ with $\bar{S}$ and $\bar{\Sigma}$. The author defines $\bar{S}$ to be a developable surface if both of the following conditions are satisfied. (1) S is a sphere. (2) The correspondences from M to $\bar{M}$ between S and $\bar{S}$ and from N to $\bar{N}$ between Σ and $\bar{\Sigma}$ are both conformal. Not every congruence of spheres (Γ) possesses the preceding properties. The author determines all such congruences (Γ), and, therefore, all conformal developables $\bar{S}$ according to his definition. Use is made of the pentasphere of Cartan. *J. DeCicco* (Chicago, Ill.).

Golifman, Roger. Sur les courbes à une dimension réelle dans l'espace hermitien hyperbolique. Bull. Soc. Roy. Sci. Liége 10, 57–67 (1941). [MF 13051]

TOPOLOGY

Šnirel'man, L. G. On certain geometrical properties of closed curves. Uspehi Matem. Nauk 10, 34–44 (1944). (Russian) [MF 12806]

[An editorial note states that the paper is a reproduction of another, under the same title, published in Sbornik Rabot Matematičeskogo Razdela Sekcii Estestvennyh i Točnyh Nauk Komakademii, Moscow, 1929, but with the earlier proofs somewhat amplified.] The most striking results of this paper are that (1) on every simple closed plane curve possessing a continuous curvature there may be found four points constituting the vertices of a square; (2) under the same hypotheses there exists on the curve a "full" system of rhombuses. A system S of rhombuses $\{R\}$ whose vertices lie on a given curve L is called "full" provided that (i) every point of L is a vertex of some R of S, (ii) each pair R and R' of S belong to a continuous one-parameter family in S and this family may be chosen so that an arbitrary vertex of R passes over to a preassigned vertex in R', (iii) none of the rhombuses is degenerate (that is, none of them is a point or doubly covered line-segment). *L. Zippin.*

Fan, Ky. À propos de la définition de connexion de Cantor. Bull. Sci. Math. (2) 68, 111–116 (1944). [MF 13250]

The Lennes-Hausdorff connectedness criterion is proved equivalent to each of four simple "chain" conditions on the space A. One of these is that, for any continuous real-valued function f defined on A, any a and b in A, and any positive d, one can find points $a=x_0, x_1, \cdots, x_n=b$, such that $|f(x_{i-1})-f(x_i)|<d$, $i=1, 2, \cdots, n$. *R. Arens.*

Balanzat, Manuel. An example of a space which is accessible, nondenumerable, separable and not perfectly separable. Revista Union Mat. Argentina 10, 163–172 (1945). (Spanish) [MF 12872]

This specimen is a T_1-space which has a countable everywhere dense subset but does not satisfy the second axiom of countability. P. Urysohn [Math. Ann. 94, 262–295 (1925), in particular, p. 288] has given an example having these properties and satisfying the Hausdorff separation axiom besides. *R. Arens* (Princeton, N. J.).

Monteiro, António Aniceto. General topology, 1. Sierpiński spaces. Cadernos de Análise Geral, no. 1. Junta de Investigação Matemática, Pôrto, 1945. 16 pp. (Portuguese) [MF 12906]

A comparison is made of various sets of postulates for Sierpiński spaces. The open sets of such spaces include the void set, the whole space, and the union of any collection of open sets. The intersection of two open sets need not be open. *R. Arens* (Princeton, N. J.).

Monteiro, António Aniceto. General topology, 2. Accessible Fréchet spaces. Cadernos de Análise Geral, no. 6. Junta de Investigação Matemática, Pôrto, 1945. 28 pp. [paged 17–44] (Portuguese) [MF 12905]

An introduction to the theory of T_1-spaces. *R. Arens* (Princeton, N. J.).

Gomes, A. Pereira. General topology, 3. Continuous functions. Cadernos de Análise Geral, no. 7. Junta de Investigação Matemática, Pôrto, 1945. 24 pp. [paged 45–68] (Portuguese) [MF 12904]

A comparison of various definitions of continuity of functions from one topological space to another. *R. Arens.*

Ferreira, Maria Helena. General topology, 4. Relativization. Cadernos de Análise Geral, no. 9. Junta de Investigação Matemática, Pôrto, 1945. 20 pp. [paged 69–88] (Portuguese) [MF 12903]

An introduction to the notion of the relative topology of a subset of a topological space. *R. Arens.*

Gomes, A. Pereira. General topology, 5. Bases and neighborhoods. Cadernos de Análise Geral, no. 10. Junta de Investigação Matemática, Pôrto, 1945. i+41 pp. [paged 89–129] (Portuguese) [MF 12902]

A general account of the subject. *R. Arens.*

Gomes, A. Pereira. General topology, 6. Compact sets. Cadernos de Análise Geral, no. 12. Junta de Investigação Matemática, Pôrto, 1945. 23 pp. (Portuguese) [MF 12901]

Various definitions of compactness are given and, in simple cases, proved equivalent. [Cf. the following review.] *R. Arens* (Princeton, N. J.).

Gomes, A. Pereira. Sur la notion d'espace compact. Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. 16, 29 pp. (1945) = Anais Fac. Ci. Pôrto 30, no. 2. (Portuguese. French summary) [MF 12928]

The author gives four definitions of compactness, meaningful in sets in which a closure operator (subject to no restrictions) can be defined. He proves their equivalence in Sierpiński spaces [cf. the sixth preceding review]. The notion of compactness is also made abstract so that it is applicable to lattices. *R. Arens* (Princeton, N. J.).

Choquet, Gustave. Étude des espaces métriques par les propriétés de leurs sous-ensembles finis. Bull. Soc. Math. France 71, 112–192 (1944). [MF 12721]

This paper investigates the topological nature of metric spaces by considerations of properties of n-tuples of points. Of special interest are the equilateral sets, the absence of which places severe restrictions upon the space. In the first of five chapters into which the paper is divided the author shows that, if E is any compact space with regular écart and θ any angle not exceeding $\pi/3$, an infinity of isosceles triangles with vertex angle θ may be inscribed in E except perhaps when E is homeomorphic to a bounded closed linear set. This theorem generalizes results due to Milgram [Proc. Nat. Acad. Sci. U. S. A. **29**, 193–195 (1943); Rep. Math. Colloquium (2) **5–6**, 25–35 (1944); these Rev. **4**, 249; **6**, 95] and the reviewer [Proc. Nat. Acad. Sci. U. S. A. **29**, 107–109 (1943); Bull. Amer. Math. Soc. **49**, 321–338 (1943); these Rev. **4**, 223, 251] which were apparently not available to the author. In particular, if E is any compact subset of the plane, an infinity of triangles similar to any given (nondegenerate) triangle may be inscribed in E except perhaps when E is a subset of an arc. Most of chapter II is devoted to obtaining the well-known Euclidean and Hilbert embedding theorems of Menger [Math. Ann. **100**, 75–163 (1928)] and the reviewer [loc. cit.]. Metric spaces with each $n+2$ points embeddable in Euclidean n-space are here called "aplatis d'ordre n," while pseudo-Euclidean $(n+3)$tuples appear as "$(n+3)$-points tordus." [The author makes no reference to the literature in which these results have long since been established.] The curvature of a triangle inscribed in a metric space E is defined by the author as the sum of the two smallest angles of the triangle; E is called flat at a point M of accumulation provided the curvatures of triangles inscribed in E approach zero when their vertices approach M. The space E is flat whenever it is flat at each of its accumulation points. Chapter III studies the topological structure of flat spaces. Utilizing results in chapters I and II it is easily shown that a flat compact metric space is the sum of a finite number of simple closed curves (pairwise disjoint) and a set which is homeomorphic to a bounded and closed linear set. Each compact and connected flat metric space has a finite length. A metric space E is semiflat at an accumulation point M whenever the smallest of the angles of triangles inscribed in E approaches zero when the vertices of the triangles approach M. The two concluding chapters study semiflat spaces, the final chapter being devoted to semiflat Cartesian arcs. The set of points at which a plane semiflat arc C possesses a tangent is seen to be everywhere dense and has the power of the continuum on every subarc of C. *L. M. Blumenthal.*

Whyburn, G. T. Uniqueness of the inverse of a transformation. Duke Math. J. 12, 317–323 (1945). [MF 12603]

Let A, B be separable metric, A a continuum, and $f(A)=B$ a continuous mapping. Denote an arbitrary point of B by y, and let E denote the set of all $f^{-1}(y)$ that are single points. (1) If f is such that the images of cut points of A are dense in B and for each y the points of $f^{-1}(y)$ are conjugate, then the set of all points y such that $f^{-1}(y)\epsilon E$ is uncountably everywhere dense in B; (2) if the cut points of A are dense in A and for each y the points of $f^{-1}(y)$ are conjugate, then E is uncountably everywhere dense in A. If f satisfies the condition that, for every y and $a, b\epsilon f^{-1}(y)$, the set of all points separating a and b in A is a subset of $f^{-1}(y)$, then f is called nonalternating in the large. A number of theorems are proved concerning mappings that are interior and nonalternating in the large; for example, (3) any cut point, end point, or simple link of A is an inverse set $f^{-1}(y)$. As a consequence of (3), such mappings satisfy the condition used in (1), that points with the same image are conjugate, and it follows from either (1) or (3) that, if the cut points of A are dense in A, any mapping of A which is interior and nonalternating in the large must be a homeomorphism.

Regarding cyclic extensibility and reducibility, it is shown that (4) if f is nonalternating in the large and A is semi-locally-connected, then in order that f be interior it is necessary and sufficient that (a) f be interior on each cyclic element of A and (b) each cut point of A be an inverse set. In conclusion the following existence theorem is proved. (5) Suppose that A is locally connected. Then there exists a nonalternating f such that B is a dendrite and all points of $f^{-1}(y)$ are conjugate; furthermore, f can be so chosen that, if C is the sum of all true cyclic elements of A, then only a countable number of points of $f(C)$ have unique inverses. As an application of (5) and (2) it is shown that, if the cut points of a locally connected continuum A are countable and dense in A, then the end points of A are uncountably everywhere dense in A. *R. L. Wilder.*

White, Paul A. New types of regular convergence. Duke Math. J. 12, 305–315 (1945). [MF 12602]

G. T. Whyburn [Fund. Math. **25**, 408–426 (1935)] defined a type of convergence which he called "regular" by approximating the condition of local connectedness in the members of the converging sequence. In this paper the condition of local connectedness is replaced by other local properties. In particular, a generalization of semilocal connectedness introduced by G. T. Whyburn [Amer. J. Math. **61**, 733–749 (1939); these Rev. **1**, 31] is used to define r-dimensional coregular convergence, and the property of complete r-avoidability introduced by R. L. Wilder [Časopis Pěst. Mat. Fys. **67**, 185–197 (1938)] is used to define r-dimensional completely avoidable regular convergence (r-c.a.-regular). A study is first made of the properties of the limit set under these types of convergence. It is shown that, if the convergence is i-regular for $i\leqq r-1$, and r-coregular, then the limit set contains no r-cut points. If the convergence is i-regular for $i\leqq r$, and r-c.a. regular, then the limit set is completely r-avoidable at each point. If the convergence is i-regular for $i\leqq n$, i-c.a. regular for $i\leqq n-2$, and $(n-1)$-coregular, then the limit of a sequence of n-dimensional closed Cantorian manifolds is an n-dimensional generalized manifold in the sense of R. L. Wilder [Ann. of Math. (2) **35**, 876–903 (1934)]. In the last section a study of the 0-dimensional convergences is made and it is shown that 0-c.a. regular convergence implies both 0-regular and 0-coregular convergence for continua. It is shown that a sequence of 2-dimensional compact manifolds converges 0-c.a. regularly to another 2-dimensional compact manifold (or a point). If each member of the sequence has n disjoint simple closed curves as boundary, then the limit set has at most n simple closed curves as its boundary. If each member of the sequence is a sphere with n handles, then the limit set is a sphere with at most n handles. *D. W. Hall.*

Hopf, Heinz. Beiträge zur Homotopietheorie. Comment. Math. Helv. 17, 307–326 (1945).

Given a connected complex K the author considers two subgroups of the nth homotopy group of K: (1) the sub-

group Γ^n of all elements homologous to zero; (2) the subgroup Π_0^n of all elements of the form $x-\alpha x$, where α is an element of the fundamental group G of K. It is proved that if K has dimension N and if the homotopy groups of K vanish for $1<n<N$ then the factor group $\Delta^N=\Gamma^N/\Pi_0^N$ is determined by G, in fact is isomorphic with the $(N+1)$th "homology" group G^{N+1} of G as defined in the author's previous paper [same Comment. **17**, 39–79 (1945); these Rev. **6**, 279]. In order to remove the condition on the dimension of K the author defines the concept of an n-dimensional "handle manifold" generalizing in an intuitive fashion the 2-dimensional sphere with handles. Cycles which are images of handle-manifolds form a subgroup P^n of the homology group B^n of K. It is then shown that if the homotopy groups of K vanish for $1<i<N$ then G^{N+1} contains a subgroup θ^N such that $G^{N+1}/\theta^N\cong\Delta^N$ and $\theta^N\cong B^{N+1}/P^{N+1}$. While proving these theorems the author also obtains alternative proofs of the main theorems of his paper mentioned above, using the concept of the "homotopy-boundary" introduced in a still earlier paper [same Comment. **14**, 257–309 (1942); these Rev. **3**, 316].

S. Eilenberg (Princeton, N. J.).

Hadwiger, H. Eine Bemerkung zum Borsukschen Antipodensatz. Vierteljschr. Naturforsch. Ges. Zürich **89**, 211–214 (1944). [MF 12240]

Proof of the theorem: if the surface of a (3-dimensional) sphere is covered by means of 3 closed sets, at least one of them is such that for every (spherical) distance d, $0<d\leqq\pi$, it contains a pair of points at distance d from one another. This is an extension of earlier results by Lusternik, Schnirelmann, Borsuk and H. Hopf. The proof, on the whole, follows elementary lines. *H. Blumberg.*

Hirsch, Guy. Sur des théorèmes de Borsuk-Ulam et de Knaster. Bull. Soc. Roy. Sci. Liége **13**, 137–145 (1944). [MF 13170]

Consider a space C with a periodic transformation A of period 2, and a mapping $T:S^{n-1}\rightarrow C$ of the $(n-1)$-sphere S^{n-1} such that $T(-x)=AT(x)$ for all $x\epsilon S^{n-1}$, where $-x$ denotes the antipodic image of x. The author proves that, if the image cycle of the fundamental $(n-1)$-cycle in S^{n-1} is homologous to zero in C (with rational coefficients), then in the various antipodic theorems of Borsuk and Lusternik-Schnirelmann the n-sphere S^n can be replaced by C. Let M^n be a closed Riemannian n-dimensional manifold with nonzero Euler characteristic and let r denote the radius of uniqueness of geodesics on M^n. Then given any $a<r$ and any mapping U of M^n into the Euclidean space R^n, two points of M^n of geodesic distance a must be identified by U.

S. Eilenberg (Princeton, N. J.).

Hirsch, Guy. Topologie. Sur un problème de H. Hopf. Bull. Soc. Roy. Sci. Liége **12**, 514–522 (1943). [MF 13154]

A proof of the following theorem is outlined. Let S^{2n} be a sphere of dimension $2n$ and let $0<a<\pi$. Let T be a mapping of S^{2n} into itself with the property that, if the spherical distance $d(x_1, x_2)$ of two points x_1 and x_2 is equal to a, then $0\neq d[T(x_1), T(x_2)]\neq\pi$. Then the degree of T is odd. The proof uses the method of sphere bundles and fibre-preserving transformations. *S. Eilenberg.*

Hirsch, Guy. Une propriété des points fixes des représentations de variétés en elles-mêmes. Bull. Sci. Math. (2) **67**, 158–168 (1943). [MF 12637]

Beweis des Satzes: "Jede stetige Abbildung einer (geschlossenen oder offenen) Mannigfaltigkeit in sich mit kompakter Fixpunktmenge kann beliebig gut durch eine stetige Abbildung mit endlicher Fixpunktmenge approximiert werden." Triangulierbarkeit der Mannigfaltigkeit wird nicht vorausgesetzt. Der Beweis beruht auf naheliegenden Modifikationen der Hilfssätze aus §2 der Arbeit des Referenten in den Math. Ann. **100**, 579–608 (1928).

H. Hopf (Zürich).

Hirsch, Guy. Sur les groupes d'homologie des espaces fibrés et des complexes de recouvrement. Bull. Soc. Roy. Sci. Liége **10**, 246–260 (1941). [MF 13064]

Given a complex C and a double covering complex $\tilde{C}$ of C the author considers a continuous mapping T of C into itself. If a certain condition is satisfied there exist two mappings $\tilde{T}_1$ and $\tilde{T}_2$ of $\tilde{C}$ into itself, covering T. The author outlines a method for deriving the homology properties (for example, the degree and the algebraic number of fixed points) of $\tilde{T}_1$ and $\tilde{T}_2$ from those of T. In the second part of the paper the author replaces $\tilde{C}$ by a sphere bundle with base space C and studies mappings $\tilde{T}$ of $\tilde{C}$ into itself that carry fibres into fibres. Again using calculations involving the degree and the number of fixed points, he states generalizations of theorems of H. Hopf and M. Rueff [Comment. Math. Helv. **11**, 49–61 (1938)]. The full exposition and proofs are to be published elsewhere. *S. Eilenberg.*

Chogoshvili, G. On duality relations in topological spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 131–132 (1945). [MF 12699]

For an arbitrary subset M of a normal space R the author defines two cohomology groups with a discrete field as coefficients. One, denoted by $\nabla_G{}^r(M)$, is called open; the other, $\nabla_F{}^r(M)$, is called closed. For the closed group, consider the totality of all closed sets F contained in M. They form a directed set by inclusion. Their cohomology groups $\nabla^r(F)$ form an inverse mapping system, the limit group of which is $\nabla_F{}^r(M)$. The open group is defined similarly using open sets G containing M. Let R be normal, locally compact, and simply connected (in the sense of homology) in dimensions r and $r+1$. The Alexander-Kolmogoroff isomorphism between the $\nabla^r(F)$ with $F\subset M$ and $\nabla^{r+1}(G)$ with $G=R-F$ implies, by way of a permanence relation, that $\nabla_F{}^r(M)$ and $\nabla_G{}^{r+1}(R-M)$ are isomorphic. There is a natural homomorphism of $\nabla_G{}^r(M)$ into $\nabla_F{}^r(M)$ for any M, induced by the natural mapping of $\nabla^r(G)$ into $\nabla^r(F)$ for any $G\supset M$ and $F\subset M$. This, coupled with the Alexander-Kolmogoroff isomorphism, determines a homomorphism t of $\nabla_F{}^r(M)$ into $\nabla_F{}^{r+1}(R-M)$; the quotient group with respect to the kernel of t and the image group can be defined independently of t. *H. Samelson* (Syracuse, N. Y.).

Pontrjagin, L. A method of calculation of homology groups. Rec. Math. [Mat. Sbornik] N.S. **11(53)**, 3–14 (1942). (English. Russian summary) [MF 12822]

The method used by the author to calculate the homology groups of compact Lie groups [same Rec. N.S. **6(48)**, 389–422 (1939); these Rev. **1**, 259] is developed and placed on a formal basis suitable for calculating the homology groups of other families of manifolds. It may be applied to a family Ω of differentiable and orientable manifolds satisfying (1) the 0-manifold consisting of a point is in Ω and (2) if $M\epsilon\Omega$ has positive dimension, then it contains two disjoint submanifolds P, $Q\epsilon\Omega$ and there is a real-valued differentiable function f defined on M which is constant and maximum on P, constant and minimum on Q, and satisfies $df\neq0$ elsewhere.

Relations among the homology groups of M, P, Q are obtained which determine the Betti numbers of M in terms of those of P and Q. The Betti numbers of all manifolds of Ω can then be determined by an inductive procedure. If the manifolds of Ω are all of even dimensions, then (1) torsion is absent so that the Betti numbers determine the homology groups and (2) all Betti numbers of odd dimensions are zero. The method is applied to obtain explicitly the Poincaré polynomials of manifolds formed from linear subspaces of a complex projective space. *N. E. Steenrod.*

MECHANICS

Hydrodynamics, Aerodynamics

Koussakov, M. Capillary-gravitational waves at the interface between two viscous liquids of finite depth. Acta Physicochimica 19, 286–294 (1944). [MF 12804]

The waves are studied under the assumption of infinitesimal wave amplitude. The undisturbed liquids are assumed to be at rest with respect to the solid boundaries. The general equations for the calculation of frequency, damping coefficient and propagation velocity are established. For very small viscosities of the liquids, it is shown that the frequency and propagation velocity are practically independent of the viscosities. However, the damping coefficient generally is a function of both the viscosities and the depth. For only one liquid, the damping coefficient of the surface wave is independent of the depth. For very viscous liquids, the motion will be aperiodic, as is to be expected.

H. S. Tsien (Pasadena, Calif.).

Grib, A. Propagation of a plane impact wave due to an ordinary explosion near a rigid wall. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 169–186 (1944). (Russian. English summary) [MF 11593]

This paper deals with the unsteady one-dimensional motion of a gas due to the decomposition of an explosive agent. The mathematical treatment consists of an application of analytical and numerical methods developed by the Russian mathematicians Christianovitch, Frankl and Kibel. The author is interested mostly in the determination of numerical parameters which depend on physical and chemical properties of the explosive and enter into the solution.

L. Bers (Syracuse, N. Y.).

Ackeret, J. Experimental and theoretical investigations of cavitation in water. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1078**, 53 pp. (8 plates) (1945). [MF 12717]

Translation of Experimentelle und Theoretische Untersuchungen über Hohlraumbildung (Kavitation) in Wasser, Promotionsarbeit, Eidgenössische Technische Hochschule in Zürich, Berlin, 1930.

Richardson, M. The pressure distribution on a body in shear flow. Quart. Appl. Math. **3**, 175–178 (1945). [MF 12654]

For the two-dimensional problem of a cylinder immersed in a shear flow, the stream function is written $\psi=\psi_0+\psi_1$, where ψ_0 corresponds to the undisturbed shear flow and ψ_1 to the disturbance flow. Hence ψ_1 is harmonic outside the cylinder. Beginning with an integral representation for ψ_1 taken from potential theory, the author deduces an integral equation for the velocity at the cylinder. In the case of vanishing vorticity, this equation reduces to Prager's [Phys. Z. **29**, 865–869 (1928)]. It is proposed to solve the equation by approximate methods to determine the velocity distribution on the cylinder. In the case of a circular cylinder in a parallel shear flow the equation can be solved explicitly; the result is shown to agree with that of Tsien [same Quart. **1**, 130–148 (1943); these Rev. **5**, 21]. *W. R. Sears.*

Jorissen, André. Conformation de l'écoulement autour d'une plaque plane placée normalement à la direction du courant. Tracé des lignes de courant et recherche de la vitesse en chaque point dans le cas d'un écoulement plan. Bull. Soc. Roy. Sci. Liége **12**, 496–506 (1943). [MF 13153]

Jacob, Caius. Sur quelques propriétés de la correspondance de M. Tchapliguine en dynamique des fluides compressibles. Bull. Math. Soc. Roumaine Sci. **42**, no. 1, 19–31 (1940). [MF 12735]

Consider two steady, irrotational, nonviscous flow fields $Z=X+iY$ and $z=x+iy$, where the first flow is compressible while the second is incompressible. Let Θ be the inclination of the velocity vector in Z with respect to OX and θ the inclination of the velocity vector in z with respect to ox. The problem is the following: what will be the most general condition on the character of the compressible fluid if $\Theta=\theta$, and the values of the stream function ψ and the potential φ are equal at two corresponding points Z and z? It is shown that three types of flow satisfy the requirements: (a) the linear pressure volume relation of Chaplygin for the compressible fluid, (b) a vortex, (c) a source. Therefore no essentially new result is obtained. The author then investigates in greater detail the geometrical properties of the flow fields satisfying the Chaplygin relation. Furthermore, the kinetic energies of the corresponding fields Z and z are proved to be equal. *H. S. Tsien* (Pasadena, Calif.).

Coburn, N. The Kármán-Tsien pressure-volume relation in the two-dimensional supersonic flow of compressible fluids. Quart. Appl. Math. **3**, 106–116 (1945). [MF 12648]

By substituting the tangent to the isentropic pressure-volume curve for the curve itself, the equations of motion for two-dimensional flows of a compressible nonviscous fluid can be reduced to the Laplace equation in the subsonic case. In the supersonic case, it is to be expected that the equations can be reduced to the simple wave equation. This is shown to be true. The solution of the flow problem then becomes very simple. For instance, when the diagonal curves of the characteristics are drawn so as to correspond to equidistant values of the arc length parameter along the characteristics, these diagonal curves will be the families of equipotentials and stream lines. Furthermore, the general representation of the stream lines depends upon two arbitrary functions. If one stream line coincides with the x-axis, these two functions are equal except for a constant. Numerical results for the velocity and density relation are given as graphs. [The graph for the example in fig. 3 is erroneous, perhaps through an error in drafting.]

H. S. Tsien (Pasadena, Calif.).

Nekrasov, A. I. Two-dimensional gas motion with subsonic velocities. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 249–266 (1944). (Russian. English summary) [MF 12225]

The treatment of subsonic compressible flows given in this paper is somewhat similar to that due to Ringleb [Z.

Angew. Math. Mech. **20**, 185–198 (1940); these Rev. **2**, 169], except that the linearization of the partial differential equation satisfied by the potential is achieved by a Legendre transformation instead of a Chaplygin transformation. Let φ denote the potential of a steady two-dimensional flow of a compressible fluid (satisfying the pressure-density relation $p\rho^{-\gamma}=\text{constant}$); let u and v be the components of the velocity vector, V the speed, a the acoustic speed and θ the angle between the velocity vector and the x-axis. The author considers the function $\Phi=ux+vy-\varphi$, not as a function of u and v but as a function of θ and τ, where

$$\tau=V/n,\quad n^2=2a_\infty^2/(\gamma-1)+V_\infty^2$$

and the subscript ∞ refers to the state of fluid at infinity. The function Φ satisfies the equation

$$(*)\qquad \tau^2(\tau^2-1)\Phi_{\tau\tau}+\tau(k^2\tau^2-1)\Phi_\tau+(k^2\tau^2-1)\Phi_{\theta\theta}=0,\quad k^2=(\gamma+1)/(\gamma-1).$$

For subsonic flows (the only case considered by the author), $\tau<1/k$. For an incompressible flow the corresponding equation reads

$$(**)\qquad \tau^2\Phi_{\tau\tau}+\tau\Phi_\tau+\Phi_{\theta\theta}=0.$$

A solution of (*) depending only on θ yields a "vortex flow"; one depending only on τ, a flow due to a "compressible source." The author investigates solutions of (*) of the form $Z_m(\tau)e^{\pm im\theta}$. They yield flows which are similar to incompressible flows given by solutions of (**) of the same form. Thus $m=\frac{1}{2}$ corresponds to a flow similar to an incompressible flow due to a dipole. In order to obtain a flow around an obstacle the author proposes an expansion of the form

$$\Phi=\Phi_0(\tau/\tau_\infty,\theta)+\tau_\infty^2\Phi_2(\tau/\tau_\infty,\theta)+\cdots,$$

where Φ_0 is the solution of (**) for the obstacle considered. As an example he treats the flow past a circular cylinder and obtains for the speed on the boundary the approximate expression

$$V=2V_\infty\cos\theta\{1-M_\infty^2\cos^2\theta/[1+\tfrac{1}{2}(\gamma-1)M_\infty^2]\}^{-1},$$

where M_∞ is the stream Mach number. *L. Bers.*

Hristianovič, S. A. The flow of gases around bodies at high, subsonic speed. Trudy Central. Aero-gidrodinam. Inst. no. 481, 52 pp. (1940). (Russian) [MF 12125]

The main problem is construction of a two-dimensional everywhere subsonic compressible potential flow past an airfoil profile. The discussion is based on the system

$$(*)\qquad \varphi_\mu=K^{\frac{1}{2}}\psi_\nu,\quad \varphi_\nu=-K^{\frac{1}{2}}\psi_\mu,$$

$$(**)\qquad \theta_\mu=s_\nu,\quad \varphi_\nu=-s_\mu.$$

Here φ is the velocity potential, ψ the stream function, θ the angle between the velocity vector and the x-axis; ν and μ are auxiliary variables; K and s are functions of the speed q which may be written in the form

$$K=(\rho_0/\rho)(1-M^2)^{\frac{1}{2}},\quad s=\int q^{-1}(1-M^2)^{\frac{1}{2}}dq,$$

where M is the local Mach number, ρ the density and ρ_0 the stagnation density. For $\theta=\mu$, $s=\nu$, equations (*) are the symmetrized Chaplygin hodograph equations [Busemann, Z. Angew. Math. Mech. **17**, 73–79 (1937); Leibenson, C. R. (Doklady) Acad. Sci. URSS (N.S.) **3**, 397–398 (1935)]. The fact that the system (*), (**) holds follows from the remark that (*) preserves its form under a conformal transformation of the (μ,ν)-plane. The author derives the system (*), (**) from a general theorem on systems of partial differential equations.

As the domain in the (μ,ν)-plane the author chooses the domain exterior to a profile $\bar{c}$ with a sharp trailing edge. He first finds the function $\bar\varphi+i\bar\psi=f(\mu+i\nu)$, the complex potential of an incompressible flow past $\bar{c}$ which satisfies the Kutta-Joukowsky condition and possesses a horizontal velocity vector of prescribed magnitude at infinity. Then he sets

$$s-i\theta=\log d(\bar\varphi+i\bar\psi)/d(\mu+i\nu),$$

so that (**) is satisfied. Now K becomes a known function of μ and ν. Next, φ and ψ are determined among those solutions of (*) which satisfy the conditions (a) $\varphi_\mu=1$, $\varphi_\nu=0$ at infinity, (b) $d\varphi/du=0$ on $\bar{c}$, (c) the circulation $\Gamma=\int_{\bar{c}}d\varphi$ has a prescribed value which is determined from the condition that the (path independent) integrals

$$(***)\qquad x=\int e^{-s}\{\cos\theta d\varphi-(\rho_0/\rho)\sin\theta d\psi\},\quad y=\int e^{-s}\{\sin\theta d\varphi+(\rho_0/\rho)\cos\theta d\psi\}$$

define a one-to-one mapping from the (μ,ν)-plane into the (x,y) plane. The mapping (***) takes $\bar{c}$ into a profile c and the domain exterior to $\bar{c}$ into the domain exterior to c; φ is the velocity potential of a compressible flow past c. The profile c can be constructed graphically and will be only slightly different from $\bar{c}$. Since s is known along $\bar{c}$, it is possible to compute the speed q along c. The essential practical difficulty consists in determining φ as a function of μ and ν. The author proposes to integrate the linear equations (*) by the well-known electrical analogy method due to Taylor [Z. Angew. Math. Mech. **10**, 334–345 (1930)].

The remaining sections contain a proof of the Kutta-Joukowski theorem for compressible fluids and an extension of the method to flows with axial symmetry. Furthermore, the author considers the simplifications resulting in the case of a thin wing (where the familiar Prandtl-Glauert formula is obtained) and in the case of a slow flow. In this last case he follows Chaplygin in setting the exponent γ in the isentropic relation equal to -1 and obtains for the profile c a formula which can be transformed into that given by Tsien [J. Aeronaut. Sci. **6**, 399–407 (1939); these Rev. **2**, 168].

[The author's method is valid only if the fact that Γ has been determined according to condition (c) implies that (α) the gradient of φ vanishes at the trailing edge of $\bar{c}$, and that (β) this gradient also vanishes at a point of $\bar{c}$ near the leading edge where $e^{-s}=0$. The author asserts that (α) can be proved. He seems to have tacitly assumed that (β) also holds. The reviewer was unable to verify either of these conditions for the case $\Gamma\neq0$. In the reviewer's opinion, the author's method must fail in the case of circulatory flows. In fact, Tsien's method for $\gamma=-1$ is essentially equivalent to the author's method and it is well known that Tsien's method does not yield circulatory flows past closed profiles.]
L. Bers (Syracuse, N. Y.).

Simonov, L. A., and Christianovitch, S. A. Influence of air compressibility on inductive velocities of an aerofoil and air-screw. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 89–98 (1944). (Russian. English summary) [MF 11599]

The authors discuss the problem of induced velocities due to airfoils (considered as lifting lines) and airscrews in the framework of the Prandtl-Glauert linearization. [For recent treatments in English see Goldstein and Young, Ministry of Aircraft Production, Aeronaut. Res. Committee, Rep. and Memoranda no. **1909** (6865), 1–20 (1943); Tsien and Lees, J. Aeronaut. Sci. 12, 173–187, 202 (1945); these Rev. 6, 193, 249.] Assuming that the velocity vector has the components $U+u'$, v, w, where U is constant and u', v, w are small of the first order, the authors write the equations of continuity and vorticity in the form

$$(1-M^2)u'_x+v_y+w_z=0, \quad \omega=\operatorname{curl}(u', v, w),$$

where M is the Mach number of the undisturbed flow. From these equations they deduce a modified Biot-Savart law: if a vortex line C_e in a nearly parallel compressible flow induces at the point (x, y, z) the velocity (u_e, v_e, w_e), and the vortex line C_i (obtained from C_e by a stretching in the x-direction in the ratio $1:(1-M^2)^{\frac{1}{2}}$) produces at the point $(x(1-M^2)^{-\frac{1}{2}}, y, z)$ the velocity (u_i, v_i, w_i), then $u_e=u_i(1-M^2)^{-\frac{1}{2}}$, $v_e=v_i$, $w_e=w_i$. This rule permits a simple discussion of the influence of compressibility on the induced velocities due to airfoils and to airscrews with an infinite number of blades as well as on wind tunnel interference. Airscrews with a finite number of blades are considered by employing cylindrical coordinates rigidly connected with the airscrew. *L. Bers* (Syracuse, N. Y.).

Tollmien, Walter. Calculation of turbulent expansion processes. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1085**, 17 pp. (2 plates) (1945). [MF 13336]

Translation of a paper in Z. Angew. Math. Mech. 6, 468–478 (1926).

Cârstoiu, I. Sur la stabilité d'une file verticale de tourbillons. Bull. Math. Soc. Roumaine Sci. **45**, 77–80 (1943). [MF 12758]

The author considers the stability of a single row of vortices composed of two component rows placed in the same line. One row consists of vortices of equal intensities Γ spaced at uniform distances h apart. The vortices in the other row are similarly spaced and have equal intensities of either Γ or $-\Gamma$, but their centers are displaced by a distance h_1 from the first row. In all the cases considered, the resultant row of vortices is found to be unstable.
C. C. Lin (Providence, R. I.).

Elasticity, Plasticity

Platrier, Charles. Au sujet d'une méthode de résolution des problèmes d'équilibre élastique. Ann. Ponts Chaussées **1940 II** (110^e année), 251–292 (1940). [MF 12881]

The author derives [without reference to the literature] the Beltrami-Michell equations of compatibility for the components of stress in an isotropic solid in equilibrium under the action of body and surface forces [see, for example, A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1927, p. 137]. He uses these equations in connection with the equations of equilibrium to set up procedures for solving the group of problems associated with the name of Saint-Venant for prisms under the action of prescribed forces and moments on their bases, and to solve a group of problems for hollow cylinders under the action of a system of prescribed internal and external pressures which may vary as linear or quadratic functions of the coordinate z measured parallel to the axis of the cylinder. *H. W. March.*

Neuber, H. Die Grundgleichungen der elastischen Stabilität in allgemeinen Koordinaten und ihre Integration. Z. Angew. Math. Mech. 23, 321–330 (1943). [MF 11747]

The fundamental equations of the theory of elastic stability in curvilinear coordinates are established by tensor methods. The author starts with the equilibrium conditions of the elements of the medium, instead of with an expression for the potential energy of the strained medium, and points out that this approach permits results which are independent of the form of the stress-strain relations. If the state of stress, the stability of which is under discussion, is homogeneous, the equations of the problem have constant coefficients and introduction of a vector stress function reduces the problem to one equation of the sixth order. For an isotropic medium reduction to equations of the second order is possible. References to and brief discussion of earlier work on the same subject are included. *E. Reissner.*

Alfrey, T. Methods of representing the properties of viscoelastic materials. Quart. Appl. Math. 3, 143–150 (1945). [MF 12650]

The author considers two mechanical interpretations of the linear viscoelastic equation (1) $PS=2Q\epsilon$, where S is the shear stress, ϵ is the shearing strain and the operators P, Q are defined in terms of the $2n$ stress-strain constants $p_{n-2}, \cdots, p_0, q_n, \cdots, q_0$ and the time parameter t by

$$P=\partial^{n-1}/\partial t^{n-1}+p_{n-2}\partial^{n-2}/\partial t^{n-2}+\cdots+p_0,$$
$$Q=q_n\partial^n/\partial t^n+q_{n-1}\partial^{n-1}/\partial t^{n-1}+\cdots+q_0.$$

The interpretations studied are: (I) a series arrangement of n Voight elements, each consisting of a spring (σ_i) and a dashpot (η_i) in parallel and satisfying

$$S=2\sigma_i\epsilon_i+2\eta_i\dot{\epsilon}_i, \qquad \epsilon=\sum\epsilon_i;$$

(II) a parallel arrangement of n Maxwell elements, each consisting of a spring and dashpot in series and satisfying

$$\dot{\epsilon}=\tfrac{1}{2}\dot{s}_i/\sigma_i+\tfrac{1}{2}s_i/\eta_i, \qquad S=\sum s_i.$$

The author shows that, for a given set of stress-strain constants of equation (1), the corresponding constants of the Voight and Maxwell arrangements are determined and conversely. From the mathematical point of view, the author's problem involves the determination of a system of n linear first order differential equations which are equivalent to a given nth order differential equation, and conversely.
N. Coburn (Austin, Tex.).

v. Mises, R. On Saint Venant's principle. Bull. Amer. Math. Soc. 51, 555–562 (1945). [MF 12817]

Expressing the opinion that A. E. H. Love's formulation of Saint Venant's principle is not very clear, the author claims that "what is meant may be correctly expressed in this way: If the forces acting upon a body are restricted to several small parts of the surface, each included in a sphere of radius ϵ, then the strains and stresses produced in the interior of the body at a finite distance from all those

parts are smaller in order of magnitude when the forces for each single part are in equilibrium than when they are not." The author proceeds to show that, in this form, Saint Venant's principle is fallacious. The counterexamples discussed in the paper deal with the stresses in an infinite half space and in a circular disk. The generalized results are: "(a) if a system of loads on an adequately supported body, all applied at surface points within a sphere of diameter ϵ, have the vector sum zero, they produce in an inner point P of the body a strain or stress value σ of the order of magnitude ϵ; (b) if the loads, in addition to having the vector sum zero, fulfill three further conditions so as to form an equilibrium system within the sphere of diameter ϵ, the σ-value produced in P will, in general, still be of the order of magnitude ϵ; (c) if the loads, in addition to being an equilibrium system, satisfy three more conditions so as to form a system in astatic equilibrium, then the σ-value produced in P will be of the order of magnitude ϵ^2 or smaller." No reference is made to other discussions of Saint Venant's principle [for instance, G. Supino, Ann. Mat. Pura Appl. (4) **9**, 91–119 (1931); O. Zanaboni, Atti Accad. Naz. Lincei, Rend. (6) **25**, 117–121, 595–601 (1937); P. Locatelli, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 502–510 (1940); **76**, 125–127 (1941); these Rev. **3**, 28]. *W. Prager.*

Pirard, A. Problème du demi-plan infini soumis à un couple incident en bordure. Bull. Soc. Roy. Sci. Liége **11**, 369–380 (1942). [MF 13110]

Pirard, A. Problème du coin plan soumis à un couple incident au sommet. Bull. Soc. Roy. Sci. Liége **11**, 547–554 (1942). [MF 13124]

Higgins, Thomas James. Analogic experimental methods in stress analysis as exemplified by Saint-Venant's torsion problem. Experimental Stress Analysis **2**, 17–27 (1945).

The author discusses the well-known analogies used in the experimental determination of the stresses in a "torsed" "prism of uniform cross section." The useful bibliography contains 74 items. *W. Prager* (Providence, R. I.).

Sen, Bibhutibhusan. Direct determination of stresses from the stress equations in some two-dimensional problems of elasticity. IV. Problems of wedges. Philos. Mag. (7) **36**, 66–72 (1945). [MF 12687]

In preceding papers [Philos. Mag. (7) **26**, 98–119 (1938); **27**, 596–604 (1939)] the author developed a "direct" method of solving two-dimensional problems of elasticity without recourse to Airy's stress function. In the present paper this method is applied to stresses in wedge-shaped isotropic and aeolotropic bodies. *W. Prager* (Providence, R. I.).

Sokolow, B. A. Problems of elastic torsion of bars. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 468–474 (1944). (Russian. English summary) [MF 12585]

The author discusses the stresses in a twisted elastic bar of varying cross-section, assuming the existence of a system of orthogonal curvilinear coordinates α_1, α_2, α_3 such that the surface of the bar is one of the coordinate surfaces $\alpha_1=$constant. In the general case, the equations become rather involved. In the case of a body of revolution [$\alpha_1=f(r, z)$], however, explicit solutions can be obtained if f_r/f_z is the product of a function of r and a function of z.

W. Prager (Providence, R. I.).

Gorgidze, A. Ja. Secondary effects in the problem of the stretching of a bar composed of different materials. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 111–114 (1943). (Russian. Georgian summary) [MF 11712]

Ruchadze, A. K. Secondary effects in the problem of bending, by a couple, of a bar composed of different materials. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 115–122 (1943). (Russian. Georgian summary) [MF 11713]

Gorgidze, A. Ja., and Ruchadze, A. K. Secondary effects in problems of stretching and bending, by a couple, of a bar composed of different materials. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **12**, 79–94 (1943). (Russian. Georgian summary) [MF 11686]

The authors investigate the problem of determining stresses and displacements in a bar consisting of several rods (with the same Poisson ratios but different elastic moduli) and considering the second power of the tension v. They solve the problem in the case of a bar under tension applied in the axial direction and in the case of bending of the bar. They obtain for the displacement vector in the first case the value $(-v\sigma x-v^2\sigma^2x,\ -v\sigma y-v^2\sigma^2y,\ vz-v^2z^2)$, where v is the tension in the z direction and σ the Poisson ratio. They obtain a more complicated expression in the case of bending. The case of a cylindrical bar which is reinforced by a cylindrical rod is discussed in detail. *S. Bergman.*

Saibel, Edward. Vibration frequencies of continuous beams. J. Aeronaut. Sci. **11**, 88–90 (1944). [MF 11764]

The author develops a method of finding the characteristic functions and frequencies for a continuous beam simply supported, clamped, or free at the ends and subjected to a number of constraints between the ends. The associated differential equation is of the form

$$(1)\quad \frac{d^2}{dx^2}\left\{EI(x)\frac{d^2Y}{dx^2}\right\}=m(x)p^2Y,\qquad y(x,t)=Y(x)\sin(pt+e),$$

which has to be solved subject to the appropriate boundary conditions and the prescribed constraints at interior points $x=c_i$, $i=1, \cdots, K$. In the absence of any constraints the problem reduces to a known boundary value problem for (1) and the author assumes that the associated closed set of characteristic functions $\phi_r(x)$ and characteristic values $\lambda_r=p_r^2$ are available numerically for small r.

The solution which also satisfies the constraints is then sought as an expansion,

$$y(x,t)=(\sum a_r\phi_r(x))\sin(\sqrt{\lambda}t+e)\equiv Y(x)\sin(\sqrt{\lambda}t+e),$$

with coefficients a_r and characteristic values λ to be determined by the condition that the total potential energy of the system should be minimal and that $Y(x)$ should satisfy the constraints. In substituting $Y(x)$ into the energy integral associated with (1), a function of the infinite set of variables a_r and λ is obtained which must be minimized under the constraints $Y(c_i)=0$. The a_r can be eliminated immediately and a frequency equation for λ is reached which, for $K=1$, is of the form

$$(2)\qquad \sum\phi_r^2/\lambda_r-\lambda=0.$$

Numerical examples (uniform cross section) are given in which quick convergence of (2) leads to a rapid and accurate determination of the λ's. Generalizations to more complex problems are discussed. *H. O. Hartley* (London).

Sergev, Sergius Ivan. The theoretical behavior and design of initially curved struts under an intermediate concentric axial load. University of Washington. Engineering Experiment Station. Bulletin no. **113**, 32 pp. (1945). [MF 12809]

Opatowski, I. Design of beams of long span and low specific strength. J. Appl. Mech. **12**, A-156–A-158 (1945). [MF 13215]

Reissner, Eric. The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. **12**, A-69–A-77 (1945). [MF 12466]

The general theory is the same as that of an earlier paper [J. Math. Phys. Mass. Inst. Tech. **23**, 184–191 (1944); these Rev. **6**, 195], which makes it possible to satisfy certain types of boundary conditions more accurately than with the classical theory. The present paper gives two applications. The first is the torsion problem for a rod of rectangular cross-section. This furnishes an opportunity for a comparison of the results obtained by use of the new theory with those obtained from the classical thin plate theory on the one hand and the exact theory of St. Venant on the other. The new theory gives results which are surprisingly accurate even in the most unfavorable case, that of a rod of square cross-section. The second case is that of an infinite plate with a circular hole. The classical theory is not applicable near the hole if the diameter of the hole is of the same order of magnitude as the thickness of the plate. Use of the new theory indicates that the classical theory underestimates the stresses at the hole by about 10 per cent even when the diameter of the hole is three times the thickness of the plate. Two cases with regard to the conditions at infinity are worked out: (a) pure bending at infinity, (b) pure twisting at infinity; in each case the edge of the hole is assumed to be free of stress. *J. J. Stoker* (New York, N. Y.).

Vlasov, V. Z. The basic differential equations in the general theory of elastic shells. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 109–140 (1944). (Russian. English summary) [MF 11601]

The paper gives a derivation of the equations for the deformation of thick elastic shells. Let α, β, γ be a system of orthogonal curvilinear coordinates, α and β being Gauss's coordinates on the middle surface and γ denoting the distance on the normal. Let u_γ denote the displacement in the direction of the normal. In the classical Kirchhoff-Love theory, u_γ depends only on α and β. The author assumes that u_γ is a linear function of γ, which requires a revision of all formulae of the theory. *A. Weinstein.*

Goldenweiser, A. L. Stressed state of a thin spherical shell. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 441–467 (1944). (Russian. English summary) [MF 12584]

This paper deals with the system of equations of the classical Kirchhoff-Love theory. It is proved that the general integral of the homogeneous system can be expressed in terms of three functions F, Φ, U. Let (α, β) be an isothermal system of coordinates on a sphere of radius R, and let $\gamma=\alpha+i\beta$. Then F and Φ are analytic functions of γ, while $U=U(\alpha, \beta)$ satisfies the differential equation $R^2\Delta U+(1+ik)U=0$, where k is a constant and Δ denotes the generalized Laplace operator corresponding to the coordinates α, β. *A. Weinstein* (Toronto, Ont.).

Dantu. Note sur l'application de la photo-élasticité à l'étude des surfaces élastiques minces. Ann. Ponts Chaussées **1941 I** (111ᵉ année), 345–360 (1941). [MF 12883]

The author establishes the equation for the stresses in a curved elastic sheet:

$$(\sigma_1-\sigma_2)(\nu_1-\nu_2)\cos 2\varphi+(\sigma_1+\sigma_2)(\nu_1+\nu_2)=2Z,$$

where σ_1, σ_2 are principal stresses; ν_1, ν_2 are principal curvatures of the surface; φ is the angle between lines of principal curvature and the directions of principal stress; Z is the normal surface load per unit of surface. By photoelastic methods, $\sigma_1-\sigma_2$ and the isostatic lines are determined and thus φ is known; the given equation then yields $\sigma_1+\sigma_2$, that is, σ_1 and σ_2. The author asserts that the application of photoelastic methods is simpler and more rapid for these curved sheets than for plane elasticity. *D. L. Holl.*

Panov, D. J. On large deflections of corrugated membranes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 226–228 (1944). [MF 12664]

To solve the problem of clamped thin corrugated circular disks under axially symmetric loads, the author employs the equations developed by von Kármán for large deflections of thin plates. The system of two nonlinear equations is modified by a term which involves the meridian curve of the corrugated surface,

$$f(r)=a\sin\tfrac{1}{2}\pi(2n+1)(R-r)/R,$$

where R is the radius of the disk or membrane, r the radial variable, and a the amplitude of the corrugated wave surface. The system is solved for the slope $w'(r)$ and the stress function $\psi(r)$ by assuming suitable expansions in terms of a small parameter $k=a/R$, with functions of r as coefficients. For $n\geqq 7$, only a few of the coefficients are required and a result is given showing the load p expressed as a cubic in the maximum deflection $W=w(0)$. *D. L. Holl* (Ames, Iowa).

Stoker, J. J. Ebene Spannungsprobleme und Differenzenrechnung. Schweiz. Arch. Angew. Wiss. Tech. **10**, 203–209 (1944). [MF 12999]

[In the original the author's name was misprinted Stocker.] The author treats the two-dimensional plane stress problem of a square plate having two corner loads applied along a diagonal of the plate and acting in the plane of the plate. The Airy stress function F satisfies $\nabla^4F=0$ and is the superposition of three functions: $F=F_1+F_2-F_3$, where F_1 and F_2 are the known stress functions for a load applied at a corner of an infinite 90° sector. To satisfy the boundary conditions the function F_3 is determined by finite differences so as to annul any stresses arising on the free edges from the superposition of F_1 and F_2 at opposite corners of the plate. A comparison is made with results obtained by M. M. Frocht for the same problem by the Ritz method as well as by photoelastic methods.

D. L. Holl (Ames, Iowa).

Shaw, F. S. The torsion of solid and hollow prisms in the elastic and plastic range by relaxation methods. Austral. Counc. Aeronaut. Rep. **ACA-11**, 38 pp. (1944). [MF 12864]

The author reviews the formulation of the torsion problem in the elastic and plastic cases of isotropic prisms and for the elastic case of orthotropic prisms (those with three mutually orthogonal planes of elastic symmetry). The elastic stress function ψ satisfies $\nabla^2\psi=-2$ with the usual conditions on the boundaries. The plastic stress function F

satisfies $\text{grad}^2 F = \text{constant}$ in the regions where the yield stress has been attained. Employing the soap film and sand hill analogies of Prandtl and Nadai, the author derives numerical solutions by finite difference equations and the relaxation method. The accuracy of the relaxation method is discussed and some comparisons are made with known solutions. In general, it seems that errors of the order of one per cent may reasonably be expected. Problems treated include a simply-connected steel joist section, a multiply-connected splined shaft, and the plastic behavior of a hollow rectangular section. No applications are made involving orthotropic prisms. *D. L. Holl* (Ames, Iowa).

Carrillo, Nabor. Differential equation of a three-dimensional plastostatic state with vertical symmetry. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario **1943**, 45–49 (1944). (Spanish) [MF 12345]

The paper considers a plastic flow in a meridian plane under the assumption that the transversal stress perpendicular to that plane has its lowest possible value. If the shear stress τ, besides fulfilling the condition of symmetry around an axis, is also independent of the distance from the axis of symmetry, then τ satisfies a nonlinear first order differential equation, which is formally solved by expanding $\tau(\varphi)$ in a series of the type $c \sin(\sum a_n \varphi^n)$, where φ is a spherical coordinate. *I. Opatowski* (Chicago, Ill.).

Ilyushin, A. A. Stability of plates and shells stressed beyond the elastic limit. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 337–360 (1944). (Russian. English summary) [MF 12388]

The buckling of plates and shells which are compressed beyond the elastic limit has hardly been investigated in spite of the fact that von Kármán's highly successful theory of the inelastic buckling of bars has long been available as a model. The reason for this is rather obvious: in the case of the inelastic buckling of bars it is sufficient to know the mechanical behavior of the material under simple compression; in the case of the inelastic buckling of plates and shells, information concerning the mechanical behavior under combined stresses is indispensable. Such information was not available at the time when the theory of the inelastic buckling of bars was developed. Since then the theory of plasticity has made considerable progress. Instead of applying its results to the inelastic buckling of plates, several authors have recommended the use of the formulas for the elastic buckling of an anisotropic plate with the bending stiffnesses αD and D in the direction of the compression and in the direction perpendicular to it, respectively, and with the torsional stiffness $\alpha^{\frac{1}{2}} D$, where α is the ratio of von Kármán's reduced modulus E_r to Young's modulus E. Similar suggestions have been made in the case of the inelastic buckling of shells. To the reviewer's knowledge, the present paper is the first serious attempt at replacing this rule-of-thumb procedure by a theory based on a thorough analysis of the stresses and strains in the buckling plate. The author's principal result is that, as a rule, the semi-empirical procedure described above underestimates the buckling load. Though this general result is probably correct, the value of all numerical results as well as of the details of the mathematical work must be questioned on account of an inconsistency in the stress-strain relations. As is necessary in the theory of plasticity, different stress-strain relations are used for "loading" and "unloading." The stress-strain relation on which all the mathematical work of the paper is based does not assure continuity of any stress component at the surface which separates the region of loading from the region of unloading [see equations (2.11) and (2.12)]. *W. Prager* (Providence, R. I.).

Sokolovsky, W. W. Elastico-plastic bending of circular plates and plane rings. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 141–166 (1944). (Russian. English summary) [MF 11602]

The paper contains a complete theory of the elastic-plastic bending of thin circular or annular plates with rotational symmetry. The usual assumption is made that the material line elements which are normal to the middle plane in the undeformed state remain normal to the middle surface of the bent plate. The stress-strain relations of Hencky and Nadai are used together with the yield condition of von Mises. The theory is worked out for perfectly plastic materials as well as for materials with strain hardening. A detailed study is made of the following three cases of loading a circular plate which is simply supported along its edge: (1) the load is uniformly distributed over the entire plate, (2) the load is uniformly distributed over the area of a circle which is concentric with the edge of the plate, (3) the load is concentrated at the center of the plate. *W. Prager* (Providence, R. I.).

Sokolovsky, W. W. The elastico-plastic equilibrium of a hollow sphere yielding the strain-hardening. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 70–78 (1944). (Russian. English summary) [MF 11470]

Applying a method similar to that used in his previous paper [same journal **7**, 273–292 (1943); these Rev. **6**, 84], the author considers a spherical shell which is supposed to be in the elastico-plastic state under uniform external and internal pressures. By symmetry, the boundary of the elastic and plastic zones is again a sphere. In the elastic zone it is assumed that Hooke's law is valid; in the plastic zone Hencky's relations (*) $2(\epsilon_r - \epsilon) = \psi(\sigma_r - \sigma)$, $2(\epsilon_\theta - \epsilon) = \psi(\sigma_\theta - \sigma)$ and R. Schmidt's conditions of strain-hardening are assumed. Here σ_r, σ_θ and ϵ_r, ϵ_θ are stress and displacement components, respectively. Furthermore, $3\epsilon = \epsilon_r + 2\epsilon_\theta$, $3\sigma = \sigma_r + 2\sigma_\theta$. On the boundary between the elastic and the plastic zones all components of stress and of deformation are supposed to be continuous. The problem can then be reduced to a system of ordinary differential equations. The author solves these equations and gives formulas for the stress components and the radial displacements in the elastic as well as in the plastic zones. All these quantities are represented in closed form as functions of ψ, introduced in (*). Three special cases are considered in detail: (a) ideal plasticity, (b) strain-hardening governed by a linear law, (c) by a power law. In these cases the expressions obtained have particularly simple forms. Some examples are given with numerical results. *S. Bergman* (Providence, R. I.).

Ishlinsky, A. J. The problem of plasticity with the axial symmetry and Brinell's test. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 201–224 (1944). (Russian. English summary) [MF 11595]

The paper is concerned with statically determinate stress distributions in plastic bodies of revolution. The principal stresses in the meridional plane are written as $\sigma - K$ and $\sigma + K$; the circumferential stress σ_θ is assumed to equal the greater of these stresses. If the angle between the direction of the principal stress $\sigma - K$ and the direction of increasing

values of r is denoted by $\alpha+\pi/4$, the stress components in the meridional plane are

$$\sigma_r=\sigma+K\sin 2\alpha,\quad \sigma_z=\sigma-K\sin 2\alpha,\quad \tau_{rz}=-K\cos 2\alpha.$$

With $\xi=(\sigma+2K\alpha)/K$, $\eta=(\sigma-2K\alpha)/K$, the equilibrium condition for these stress components can be written in the form

$$(1)\qquad \partial\xi/\partial s=\partial\eta/\partial s'=(\cos\alpha-\sin\alpha)/r,$$

where $\partial/\partial s$ and $\partial/\partial s'$ denote differentiation in the directions of the slip lines which form the angles α and $\alpha+\pi/2$ with the direction of increasing values of r. If ξ and η, and hence $\alpha=(\xi-\eta)/4$, are known along a line L which is not a characteristic (Cauchy problem), the following approximate method of integration is used: through adjacent points A and B of L straight lines of the directions α_A and $\alpha_B+\pi/2$ are drawn; if C is their point of intersection, the segments AC and BC are taken as approximations to the slip lines through A and B; integration of (1) along these segments leads to the values of ξ and η, and hence of α, at C; etc. This procedure can be adapted to the Goursat problem and to the mixed problem. The pressing of a rigid sphere into a plastic half space is treated as an example. The results are of interest in connection with a theoretical discussion of the Brinell test. *W. Prager* (Providence, R. I.).

Kapuano, I. Sur les réseaux de Hencky-Prandtl. Rev. Fac. Sci. Univ. Istanbul (A) 9, 35–60 (1944). (French. Turkish summary) [MF 12242]

This paper contains a contribution to the geometry of the orthogonal nets formed by the curves of maximum shearing stress (slip lines) in the plane plasticity problem. Such nets are called Hencky-Prandtl (H.P.) nets. In an introductory chapter, a review is given of some well-known properties of H.P. nets due to Hencky, Prandtl, Carathéodory-Schmidt, and the author. In chapter 1 the author defines points of Cesàro for these nets, proves a theorem involving them, and furnishes some applications. A point of Cesàro is defined as follows: let C_0 be the center of curvature of a curve Γ of the net at the point P; let C_1 be the center of curvature of the first evolute of Γ at C_0, etc.; the limit point C (if it exists) of the points $C_0, C_1, C_2, \cdots$ is the desired point. The author proves that, if one curve of an H.P. net possesses a point of Cesàro, then every curve of the family to which this curve belongs possesses the same point of Cesàro (within the domain in which these curves are analytic). In chapter 2 the author proves various theorems on approximations. For instance, two orthogonal curves of one H.P. net are assumed to approximate to two orthogonal curves of another H.P. net and bounds are determined for the approximation of one net to the other at an arbitrary point. Chapter 3 is concerned with nets whose curves intersect at an angle $\frac{1}{2}\pi-\rho$, where ρ is constant, and possess the Hencky property. The author states that such nets are slip lines for a properly chosen yield condition. The following generalization of the author's theorem for H.P. nets is proved: if a net R consists of two families of curves which intersect at the angle $\frac{1}{2}\pi-\rho$, and if R possesses the Hencky property, then the net, one of whose families consists of the evolutes to one family of R and the other of whose families cuts the evolutes under the angle $\frac{1}{2}\pi-\rho$, possesses the Hencky property. The general equation of R is determined and a generalization of the Prandtl property is derived. *N. Coburn* (Austin, Tex.).

Sedgewick, C. H. W. On plastic bodies with rotational symmetry. Quart. Appl. Math. 3, 178–182 (1945). [MF 12655]

This note deals with the problem of rotationally symmetric states of plastic stress. A system of orthogonal coordinates α, β, θ is chosen, where $\alpha=\alpha(r,z)$, $\beta=\beta(r,z)$ and the linear element is $ds^2=A^2d\alpha^2+B^2d\beta^2+r^2d\theta^2$. The first result obtained is the fact that, when the lines $\alpha=$constant and $\beta=$constant are lines of principal stress, the relation $ABr=1$ may be made to hold by the choice of proper scales along these lines. This is analogous to a known result in plane strain. The second result is that networks of cycloids or logarithmic spirals are possible solutions for the lines of maximum shearing stress. *E. Reissner.*

Deutsch, Walther. On the pseudo-plastic state. Philos. Mag. (7) 36, 115–121 (1945). [MF 13278]

The flow of a viscous substance under shear is pictured as a sequence of jerks rather than a continuous process, each jerk consisting of a sudden small increase in stress followed by a relaxation which restores the original stress. The formulas developed on the basis of this picture show a decrease of the apparent viscosity with increasing rate of shear, in good qualitative agreement with experiments. *W. Prager* (Providence, R. I.).

Reiner, M. A mathematical theory of dilatancy. Amer. J. Math. 67, 350–362 (1945). [MF 12916]

The paper is concerned with the most general form which the relation between the tensors of viscous stress and velocity strain can assume in an isotropic viscous fluid. In the first part the author assumes the existence of a dissipation function such that any component of the viscous stress equals the derivative of the dissipation function with respect to the corresponding component of the velocity strain. This assumption is dropped in the second part, and the viscous stress is written as a polynomial in the velocity strain. By means of the Cayley-Hamilton equation this polynomial can be reduced to three terms, namely the unit tensor, the tensor of velocity strain e_{ik}, and its square $e_{ij}e_{jk}$, each multiplied by some invariant of e_{ik}. The conditions for the existence of a dissipation function are established. Finally, the velocity strain is expressed in terms of the viscous stress, and it is shown that a tensor of viscous stress with vanishing trace may well produce a cubic dilatation. This phenomenon has been described as dilatancy by O. Reynolds. *W. Prager* (Providence, R. I.).

BIBLIOGRAPHICAL NOTE

Uspehi Matematičeskih Nauk 10 (1944).

This volume contains collections of papers on "visual geometry" and on probability and mathematical statistics. The first part contains translations of papers by A. Cauchy [J. École Polytech. 9, 68–86 (1813)=Oeuvres Complètes, ser. 2, vol. 1, Paris, 1905, pp. 7–25], A. Cayley [Philos. Mag. (4) 17, 123–128 (1859)=Collected Mathematical Papers, vol. 4, Cambridge, 1891, pp. 81–85], W. L. Stringham [Amer. J. Math. 3, 1–14 (1880)] and H. Hopf [Comment. Math. Helv. 9, 303–319 (1937)] and the papers by Dukor, Šnirel'man and Šklyarskiĭ reviewed in these Rev. 6, 280; 7, 35, 12. The second part contains translations of papers by S. Bernstein [Math. Ann. 97, 1–59 (1926)], H. Cramér [Actualités Sci. Ind., no. 736, 5–23 (1938)], J. Neyman [Actualités Sci. Ind., no. 739, 25–57 (1938)] and the papers of Gnedenko and Smirnov reviewed in these Rev. 7, 19.

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